ON GOWDY VACUUM SPACETIMES

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Abstract. By Fuchsian techniques, a large family of Gowdy vacuum spacetimes have been constructed for which one has detailed control over the asymptotic behaviour. In this paper we formulate a condition on initial data yielding the same form of asymptotics.

1. Introduction

This paper is concerned with the study of cosmological singularities. By a cosmological spacetime we mean a globally hyperbolic Lorentz manifold with compact spatial Cauchy surfaces satisfying Einstein's equations. A singularity is characterized by causal geodesic incompleteness (assuming the spacetime satisfies some natural maximality condition). Causal geodesic incompleteness, and thus singularities, is guaranteed in general situations by the singularity theorems. However, the question of curvature blow up at the singularity, and the related question of strong cosmic censorship are a separate issue. The desire to answer these questions motivated this paper.

Most of the work in the area of cosmological singularities has concerned the spatially homogeneous case. However, some classes of spatially inhomogeneous spacetimes have been studied analytically and numerically. In particular, the so called Gowdy spacetimes have received considerable attention. The reason for this is probably the fact that analyzing the Gowdy spacetimes is on the borderline of what is doable and what is not. These spacetimes were first introduced in [5] (see also [3]), and in [9] the basic questions concerning global existence were answered. We will take the Gowdy vacuum spacetimes on $\mathbb{R} \times T^3$ to be metrics of the form (1.1). However, some sort of motivation for this choice seems to be in order. Below, we give a rough description of more natural conditions that lead to this form of metric. In fact, the conditions below do not imply the form (1.1), see [3] pp. 116-117. However, the discrepancy can be eliminated by a coordinate transformation which is local in space. Combining this observation with domain of dependence arguments hopefully convinces the reader that nothing essential is lost by considering metrics of the form (1.1). The description below is brief and we refer the interested reader to [5] and [3] for more details. The following conditions can be used to define the Gowdy vacuum spacetimes:

- It is an orientable globally hyperbolic vacuum spacetime.
- It has compact spatial Cauchy surfaces.
- There is a smooth effective group action of $U(1)\times U(1)$ on the Cauchy surfaces under which the metric is invariant.
- The twist constants vanish.

Let us explain the terminology. A group action of a Lie group G on a manifold M is effective if $gp = p$ for all $p \in M$ implies $g = e$. Due to the existence of the symmetries we get two Killing fields. Let us call them X and Y. The twist constants are defined by

$$\kappa_X = \epsilon_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma X^\delta \quad \text{and} \quad \kappa_Y = \epsilon_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma Y^\delta.$$

The fact that they are constants is due to the field equations. By the existence of the effective group action, one can draw the conclusion that the spatial Cauchy surfaces have topology T^3, S^3, $S^2 \times S^1$ or a Lens space. In all the cases except T^3, the twist constants have to vanish. However, in the case of T^3 this need not be true, and the condition that they vanish is the most unnatural of the ones on the list above. There is however a reason for separating the two cases. Considering the case of T^3 spatial Cauchy surfaces, numerical studies indicate that the Gowdy case is convergent [2] and the general case is oscillatory [1]. Analytically analyzing the case with non-zero twist constants can therefore reasonably be expected to be significantly more difficult than the Gowdy case. In this paper we will consider the Gowdy T^3 case.

Due to the numerical studies, cf. [2], the picture as to what should happen is quite clear. In order to formulate the conclusions, we need to parametrize the metric. One way of doing so is

$$g = e^{(\tau-\lambda)/2}(-e^{-2\tau}d\tau^2 + d\theta^2) + e^{-\tau}[e^P d\sigma^2 + 2e^P Q d\sigma d\delta + (e^P Q^2 + e^{-P})d\delta^2]. \tag{1.1}$$

Here, $\tau \in \mathbb{R}$ and (θ, σ, δ) are coordinates on T^3. The evolution equations become

$$P_{\tau\tau} - e^{-2\tau}P_{\theta\theta} - e^{2P}(Q_\tau^2 - e^{-2\tau}Q_\theta^2) = 0 \tag{1.2}$$

$$Q_{\tau\tau} - e^{-2\tau}Q_{\theta\theta} + 2(P_\tau Q_\tau - e^{-2\tau}P_\theta Q_\theta) = 0, \tag{1.3}$$

and the constraints

$$\lambda_\tau = P_\tau^2 + e^{-2\tau}P_\theta^2 + e^{2P}(Q_\tau^2 + e^{-2\tau}Q_\theta^2) \tag{1.4}$$

$$\lambda_\theta = 2(P_\theta P_\tau + e^{2P}Q_\theta Q_\tau). \tag{1.5}$$

Obviously, the constraints are decoupled from the evolution equations, excepting the condition on P and Q implied by (1.5). Thus the equations of interest are the two non-linear coupled wave equations (1.2)-(1.3). In this parametrization, the singularity corresponds to $\tau \to \infty$, and the subject of this article is the asymptotics of solutions to (1.2)-(1.3) as $\tau \to \infty$. The asymptotics we derive will then be used to obtain conclusions concerning curvature blow up. There is a special case of these equations determined by the condition $Q = 0$ which is called the polarized case. This has been handled in [7], which also considers the other topologies. The asymptotic behaviour of the solution P to (1.2) in the polarized case is given by

$$P(\tau, \theta) = v(\theta)\tau + \phi(\theta) + e^{-\epsilon\tau}u(\tau, \theta)$$

where $\epsilon > 0$ and $u(\tau, \theta) \to 0$ as $\tau \to 0$. In this situation, v and ϕ are arbitrary smooth functions on the circle. In the general case, the numerical studies indicate that the "velocity" v should typically be confined to the open interval $(0, 1)$. To be more precise, the following asymptotics are expected in general:

$$P(\tau, \theta) = v(\theta)\tau + \phi(\theta) + e^{-\epsilon\tau}u(\theta, \tau) \tag{1.6}$$

$$Q(\tau, \theta) = q(\theta) + e^{-2v(\theta)\tau}[\psi(\theta) + w(\tau, \theta)] \tag{1.7}$$

where $\epsilon > 0$ and $w, u \to 0$ as $\tau \to \infty$ and $0 < v(\theta) < 1$. A heuristic argument motivating the condition on the velocity can be found in [2]. However, the numerical simulations also indicate the occurrences of "spikes". Let us describe one sort of spike that can occur. It can happen that at a spatial point θ_0, P_τ will, in the limit, have a value greater than 1 whereas the limiting values of P_τ in a punctured neighbourhood of θ_0, P_τ belong to $(0,1)$. Furthermore, Q converges nicely in a neighbourhood of θ_0 with a zero of the spatial derivative of Q at θ_0. Beyond the numerical indications of these types of features, a family of solutions with spikes have been constructed in [11], so that the behaviour described above is known to occur. The type of spike described above is a "true" spike. There are also other types of spikes called "false" spikes at which Q has a discontinuity. We refer the reader to [11] for more details. One relevant question to ask is whether the spikes are a result of a bad parametrization of the metric or if they really have a geometrical significance. It seems that the false spikes are a result of bad parametrization, but the true spikes can be detected by curvature invariants [11].

In this paper we will not be concerned with spikes, but will focus on solutions with an asymptotic behaviour of the form (1.6)-(1.7). By the so called Fuchsian techniques one can construct a large family of solutions with such asymptotic behaviour. In fact, given functions v, ϕ, q and ψ from S^1 to $\mathbb{R}$ of a suitable degree of smoothness and subject to the condition $0 < v < 1$, one can construct solutions to (1.2)-(1.3) with asymptotics of the form (1.6)-(1.7). The proof of this in the real analytic case can be found in [8] and [10] covers the smooth case. One nice feature of this construction is the fact that one gets to specify four functions freely, just as as if though one were specifying initial data for (1.2)-(1.3).

The purpose of this paper is to provide a condition on the initial data yielding the asymptotic behaviour (1.6)-(1.7). There are several reasons for wanting to prove such a statement. As was mentioned above, one can construct a large family of solutions with the desired asymptotic behaviour, but it is not clear how big this family is in terms of initial data. In this paper we prove the existence of an open set of initial data yielding the desired asymptotics. Observe that the equations (1.2)-(1.3) are not time translation invariant, so that at which time one starts is of relevance. The open set in the initial data will thus depend on the starting time τ_0. The condition demanded in this paper is not only sufficient, but in fact also necessary for obtaining asymptotics of the form (1.6)-(1.7) in the sense that, if a solution has this form of asymptotics, then for a late enough time, the condition on the initial data will be satisfied. In this sense, the condition described in this article is a characterization of the solutions with asymptotic behaviour (1.6)-(1.7) in terms of initial data. Observe finally that the condition, even though it is formulated as a global condition on all of S^1 in this paper, can be applied locally due to domain of dependence arguments. Thus the condition prescribed here should be of relevance, and should in fact be applicable in a neighbourhood of almost all spatial points, even in the case with spikes. The problem in the general case of course being that of proving that the evolution takes you to such a region. Finally, let us observe that this is not the first result in this direction. In [4], Chruściel considers developments of perturbations of initial data for the Kasner $(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ metric within the Gowdy class. He proves, among other things, curvature blow up for these developments. In our setting the Kasner $(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ metrics correspond to $P = Q = 0$.

2. The equations and an outline of the argument

Since the method does not depend on the dimension, and since the arguments in a sense become more transparent in a more general setting, we will consider the following equations on $\mathbb{R} \times T^d$:

$$\begin{aligned} P_{\tau\tau} - e^{-2\tau}\Delta P - e^{2P}(Q_\tau^2 - e^{-2\tau}|\nabla Q|^2) &= 0 && (2.1)\\ P(\tau_0,\cdot) = p_0, \quad P_\tau(\tau_0,\cdot) &= p_1 \\ Q_{\tau\tau} - e^{-2\tau}\Delta Q + 2(P_\tau Q_\tau - e^{-2\tau}\nabla P\cdot\nabla Q) &= 0 && (2.2)\\ Q(\tau_0,\cdot) = q_0, \quad Q_\tau(\tau_0,\cdot) &= q_1. \end{aligned}$$

These equations have some similarities with wave maps. Let

$$g = -dt\otimes dt + \sum_{i=1}^d dx^i \otimes dx^i$$

be the Minkowski metric on $\mathbb{R}\times T^d$ and let

$$g_0 = dP\otimes dP + e^{2P} dQ\otimes dQ$$

be a metric on $\mathbb{R}^2$. Observe that $(\mathbb{R}^2, g_0)$ is isometric to hyperbolic space. The wave map equations for a map from $(\mathbb{R}\times T^d, g)$ to $(\mathbb{R}^2, g_0)$ is given by the Euler-Lagrange equations corresponding to the action

$$\int g_{0,ab}g^{\alpha\beta}\partial_\alpha u^a \partial_\beta u^b dtdx = \int [-P_t^2 + |\nabla P|^2 - e^{2P}Q_t^2 + e^{2P}|\nabla Q|^2]dxdt.$$

This should be compared with (2.1) and (2.2) which are obtained as the Euler-Lagrange equations corresponding to the action

$$\int [-P_\tau^2 + e^{-2\tau}|\nabla P|^2 - e^{2P}Q_\tau^2 + e^{2P-2\tau}|\nabla Q|^2]dxd\tau.$$

The exact statement of the result can be found in section 9. It is however rather lengthy, and therefore we wish to state a somewhat less technical consequence here. We need to define some energy norms, but first note the following convention concerning multi-indices. If $\alpha = (\alpha_1, ..., \alpha_d)$ where α_i are non-negative integers, then

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1}\cdots \partial x_d^{\alpha_d}},$$

where $|\alpha| = \alpha_1 + ... + \alpha_d$. The natural energy norm for P is

$$\mathcal{E}_k(\tau) = \mathcal{E}_k(P,\tau) = \frac{1}{2}\sum_{|\alpha|=k}\int_{T^d}[(D^\alpha\partial_\tau P)^2 + e^{-2\tau}|\nabla D^\alpha P|^2]d\theta \tag{2.3}$$

and the one for Q is

$$E_k(\tau) = E_k(P,Q,\tau) = \frac{1}{2}\sum_{|\alpha|=k}\int_{T^d}[e^{2P}(D^\alpha\partial_\tau Q)^2 + e^{2P-2\tau}|\nabla D^\alpha Q|^2]d\theta. \tag{2.4}$$

Finally it is natural to introduce

$$\mathcal{F}_k(\tau) = \mathcal{F}_k(P,\tau) = \frac{1}{2}\sum_{|\alpha|=k}\int_{T^d}|\nabla D^\alpha P|^2 d\theta.$$

If $p_0, p_1, q_0, q_1 \in C^\infty(T^d, \mathbb{R})$, we will by $\epsilon_k(p_0, p_1, \tau)$ mean $\mathcal{E}_k$ with P replaced by p_0 and P_τ replaced by p_1. We associate $e_k(p_0, q_0, q_1, \tau)$ with E_k and $\nu_k(p_0)$ with $\mathcal{F}_k$ similarly.

Theorem 2.1. *Let $p_0, p_1 \in C^\infty(T^d, \mathbb{R})$ satisfy*

$$2\gamma \leq p_1 \leq 1 - 2\gamma,$$

where $\gamma > 0$. Then, if τ_0 is big enough and $e_k(p_0, q_0, q_1, \tau_0)$, $k = 0, ..., m_d = 2[d/2]+2$ are small enough, we have smooth solutions to (2.1) and (2.2) on $[\tau_0, \infty) \times T^d$ with the following properties: there are $v, w, q \in C^\infty(T^d, \mathbb{R})$ with

$$\gamma \leq v \leq 1 - \gamma$$

and polynomials $\pi_{1,k}, \pi_{2,k}$ in $\tau - \tau_0$ for every non-negative integer k such that

$$\|P - \rho\|_{C^k(T^d,\mathbb{R})} \leq \pi_{1,k} \exp[-2\gamma(\tau - \tau_0)],$$

$$\sum_{|\alpha| \leq k} \|e^{2\rho}[D^\alpha Q - D^\alpha q]\|_{C(T^d,\mathbb{R})} \leq \pi_{2,k},$$

where $\rho = v \cdot (\tau - \tau_0) + w$.

Remark. The sizes of τ_0 and the $e_k(p_0, q_0, q_1, \tau_0)$ only depend on γ, $\epsilon_k(p_0, p_1, \tau_0)$ and $\nu_k(p_0)$, $k = 0, ..., m_d$. The reader interested in more conclusions is referred to section 9.

Let us give an outline of the proof. Given a smooth function P, we can view (2.2) as a linear equation for Q. The central part of the argument is an analysis of this linear equation assuming P satisfies

$$0 < \gamma \leq P_\tau \leq 1 - \gamma$$

and $\mathcal{E}_k(P)$ bounded for $k = 0, ..., m_d = 2[d/2] + 2$. Under these conditions on P one can prove that E_k satisfies

$$E_k^{1/2}(\tau) \leq \mathcal{P}_k(\tau - \tau_0) \exp[-\gamma(\tau - \tau_0)] \tag{2.5}$$

for $k = 0, ..., m_d$. Here $\mathcal{P}_k$ is a polynomial in $\tau - \tau_0$ whose coefficients depend on the values of E_k, $\mathcal{E}_k$ and $\mathcal{F}_k$ at $\tau = \tau_0$ for $k = 0, ..., m_d$. The polynomials $\mathcal{P}_k$ have one important property. If one lets $E_k(\tau_0)$ go to zero, then the coefficients of the polynomial go to zero, assuming the other parameters are constant. In consequence, under these conditions on P, one has very good control of Q. The idea is then to consider the following iteration:

$$P_{n,\tau\tau} - e^{-2\tau}\Delta P_n - e^{2P_{n-1}}(Q_{n,\tau}^2 - e^{-2\tau}|\nabla Q_n|^2) = 0 \tag{2.6}$$

$$Q_{n,\tau\tau} - e^{-2\tau}\Delta Q_n + 2(P_{n-1,\tau}Q_{n,\tau} - e^{-2\tau}\nabla P_{n-1} \cdot \nabla Q_n) = 0 \tag{2.7}$$

$$P_{0,\tau\tau} - e^{-2\tau}\Delta P_0 = 0 \tag{2.8}$$

where

$$P_n(\tau_0, \cdot) = p_0,\ Q_n(\tau_0, \cdot) = q_0,\ P_{n,\tau}(\tau_0, \cdot) = p_1,\ Q_{n,\tau}(\tau_0, \cdot) = q_1.$$

Observe that we only need to solve linear equations during the iteration in that if P_{n-1} is given, then we can compute Q_n using (2.7), so that (2.6) also becomes a linear equation. One sets up an induction hypothesis on P_n amounting to the statements

$$0 < \gamma \leq P_{n,\tau} \leq 1 - \gamma \quad \text{and} \quad \mathcal{E}_k(P_n, \tau) \leq c_k < \infty, \quad k = 0, ..., m_d. \tag{2.9}$$

It is not too difficult finding conditions on the initial data of P ensuring that P_0 satisfies these conditions. By the above observations concerning the linear equation (2.7), one gets very good control of the behaviour of Q_{n+1} if (2.9) holds. Inserting this information into (2.6), it turns out that the propagation of the inductive hypotheses essentially boils down to a smallness condition on $E_k(\tau_0)$ $k = 0, ..., m_d$ due to the structure (2.5). In this way, we produce a sequence of iterates obeying (2.9). The arguments proving convergence turn out to be similar to the arguments proving the propagation of the inductive hypotheses, and a smallness condition on $E_k(\tau_0)$ ensures the desired behaviour. Thus one produces a solution to (2.1) and (2.2) with certain properties. Due to the fact that one knows the solution to have these extra properties, one can show that it has the desired asymptotic behaviour.

3. Local existence

Let us here state the local existence result we will need. We are interested in equations of the form

$$\begin{aligned} \Box \mathbf{P}(t,x) &= \mathbf{F}(t,x,\mathbf{P},\mathbf{P}_t,\nabla \mathbf{P}) \\ \mathbf{P}(t_0,\cdot) &= \mathbf{p}_0, \quad \mathbf{P}_t(t_0,\cdot) = \mathbf{p}_1. \end{aligned} \tag{3.1}$$

Proposition 3.1. *Consider the equation (3.1) where $t_0 \in \mathbb{R}_-$, $(t,x) \in \mathbb{R}_- \times T^d$, $(\mathbf{p}_0, \mathbf{p}_1) \in H^{k+1}(T^d,\mathbb{R}^l) \times H^k(T^d,\mathbb{R}^l)$, $\mathbf{F}$ is a smooth function and $k \geq m_d/2 = [d/2]+1$. Here, $\mathbb{R}_- = (-\infty, 0)$. Then there are $T_1, T_2 \in \mathbb{R}_-$ with $T_1 < t_0 < T_2$ such that there is a unique solution solution of (3.1) in*

$$C(I, H^{k+1}(T^d,\mathbb{R}^l)) \cap C^1(I, H^k(T^d,\mathbb{R}^l)) \tag{3.2}$$

where $I = [T_1, T_2]$. Let $T_{\max}$ be the supremum of the times $T \in \mathbb{R}_-$ such that there is a unique solution to (3.1) in (3.2) for $I = [t_0, T]$ and define $T_{\min}$ similarly. Then either $T_{\max} = 0$, or

$$\sup_{t \in [t_0, T_{\max})} \|\mathbf{P}\|_{C^1([0,t]\times T^d,\mathbb{R}^l)} = \infty.$$

The statement for $T_{\min}$ is similar.

Remark. The fact that T^d is compact makes some of the usual conditions on $\mathbf{F}$ unnecessary.

The proof uses estimates of the form (6.4.5)' of [6] adapted to the torus case. A similar result for $k \geq m_d$ only requires Sobolev embedding and is sufficient for our purposes. We will later solve the non-linear problem by carrying out an iteration, and it will then be of interest to solve equations of the form

$$\begin{aligned} Q_{tt} - \Delta Q = G_1 Q_t + G_2 \cdot \nabla Q + G_3 \\ Q(t_0,\cdot) = q_0 \quad Q_t(t_0,\cdot) = q_1 \end{aligned}$$

where $G_1, G_2, G_3 \in C^\infty(\mathbb{R}_- \times T^d, \mathbb{R})$ and the initial data are smooth. Observe that local as well as global existence of smooth solutions to this problem is assured by Proposition 3.1. Observe finally that equations of the form

$$\mathbf{P}_{\tau\tau}(\tau,x) - e^{-2\tau}\Delta \mathbf{P}(\tau,x) = \mathbf{F}(\tau,x,\mathbf{P},\mathbf{P}_\tau,\nabla \mathbf{P})$$

on $\mathbb{R} \times T^d$ can be transformed to equations of the form (3.1) by the transformation $t(\tau) = -e^{-\tau}$.

4. The polarized case

It is always instructive to start with an easier subcase, and in this case we have the added incentive that the zeroth iterate of the iteration belongs to this subclass, so let us consider the polarized case. Let P solve

$$P_{\tau\tau} - e^{-2\tau}\Delta P = 0. \tag{4.1}$$

Proposition 4.1. *Consider a smooth solution P to the polarized equation (4.1). Then there are $v, w \in C^\infty(T^d, \mathbb{R})$ and positive constants $C_k \in \mathbb{R}$ for every non-negative integer k such that*

$$\|P_\tau - v\|_{C^k(T^d,\mathbb{R})} + \|P - v\tau - w\|_{C^k(T^d,\mathbb{R})} \leq C_k(1+\tau)e^{-2\tau} \tag{4.2}$$

for all $\tau \geq 0$.

Proof. The energy $\mathcal{E}_k$ defined by (2.3) satisfies

$$\frac{d\mathcal{E}_k}{d\tau} \leq 0. \tag{4.3}$$

By Sobolv embedding, we conclude that

$$\sum_{|\alpha|=k} [\|D^\alpha \partial_\tau P\|_{C(T^d,\mathbb{R})} + e^{-\tau}\|\nabla D^\alpha P\|_{C(T^d,\mathbb{R})}] \leq C_k < \infty$$

for all k. Inserting this information into (4.1), we conclude that

$$\sum_{|\alpha|=k} \|D^\alpha \partial_\tau^2 P\|_{C(T^d,\mathbb{R})} \leq C_k e^{-\tau}$$

for all k and α. We conclude the existence of a function $v \in C^\infty(T^d, \mathbb{R})$ such that

$$\|P_\tau - v\|_{C^k(T^d,\mathbb{R})} \leq C_k e^{-\tau},$$

which in its turn proves the existence of a $w \in C^\infty(T^d, \mathbb{R})$ such that

$$\|P - v\tau - w\|_{C^k(T^d,\mathbb{R})} \leq C_k e^{-\tau}.$$

One consequence of this is of course that

$$\|P\|_{C^k(T^d,\mathbb{R})} \leq C_k(1+\tau)$$

for $\tau \geq 0$. Inserting this in (4.1) and going through the same steps as above, one ends up with (4.2). □

Observe that in the end, it turns out that P and all its spatial derivatives do not grow faster than linearly. However, the natural consequence of the boundedness of $\mathcal{E}_k$ is that expressions of the form

$$\frac{1}{2}\sum_{|\alpha|=k} \int_{T^d} e^{-2\tau}|\nabla D^\alpha P|^2 d\theta$$

are bounded. In other words, the form of the energy (2.3), forced upon us by the energy methods, is not well suited to the behaviour of the solutions. The way we achieved the linear growth estimate in the proposition above, was through a procedure which was very wasteful of derivatives. This is not likely to be successful in the general non-linear case. However, there is another point of view, and we will describe it in the next section.

5. Energies

In this section we gather some observations concerning the type of energies we will be using. Let $\tau_0 \in \mathbb{R}_+ = [0, \infty)$, $I = [\tau_0, \infty)$ and let $\psi \in C^\infty(\mathbb{R} \times T^d, \mathbb{R})$. Let

$$\mathcal{F}_k(\psi, \tau) = \frac{1}{2} \sum_{|\alpha|=k} \int_{T^d} |\nabla D^\alpha \psi|^2 d\theta. \tag{5.1}$$

Lemma 5.1. *Assume $\psi \in C^\infty(\mathbb{R} \times T^d, \mathbb{R})$ satisfies*

$$\mathcal{E}_k^{1/2}(\psi, \tau) \leq \epsilon_k < \infty$$

for $\tau \in I$ and $k = 0, ..., m$, where $\mathcal{E}_k$ is defined in (2.3), $m \geq m_d/2$, $m_d = 2[d/2] + 2$ and the ϵ_k are constants. Then, for $k \leq m - m_d/2$,

$$\sum_{|\alpha|=k} [\|D^\alpha \partial_\tau \psi(\tau, \cdot)\|_{C(T^d,\mathbb{R})} + e^{-\tau} \|\nabla D^\alpha \psi(\tau, \cdot)\|_{C(T^d,\mathbb{R})}] \leq C(\epsilon_k + \epsilon_{k+m_d/2})$$

for $\tau \in I$, where the ϵ_k may be omitted if $k > 0$. Furthermore,

$$\mathcal{F}_k^{1/2}(\psi, \tau) \leq C[\mathcal{F}_k^{1/2}(\psi, \tau_0) + \epsilon_{k+1}(\tau - \tau_0)] \tag{5.2}$$

if $k \leq m - 1$, so that

$$\sum_{|\alpha|=k} \|\nabla D^\alpha \psi\|_{C(T^d,\mathbb{R})} \leq C[\mathcal{F}_{k+m_d/2}^{1/2}(\psi, \tau_0) + \epsilon_{k+1+m_d/2}(\tau - \tau_0)]$$

if $k \leq m - m_d/2 - 1$.

Remark. When we write $\|f\|_{C(T^d,\mathbb{R})}$ for a vector valued function f, we mean the sup norm of the Euclidean norm of the function.

Proof. The first inequality follows from Sobolev embedding, as well as the third, given the second. The second is proved by computing

$$\frac{d\mathcal{F}_k}{d\tau} = \sum_{|\alpha|=k} \int_{T^d} \nabla D^\alpha \psi \cdot \nabla D^\alpha \partial_\tau \psi d\theta \leq 2C\mathcal{F}_k^{1/2} \mathcal{E}_{k+1}^{1/2}.$$

□

Observe that the main point of this lemma is the estimate (5.2). We get a linear growth estimate for $\mathcal{F}_k^{1/2}(\psi, \tau)$ if we know that $\mathcal{E}_{k+1}(\psi, \tau)$ is bounded for $\tau \in I$. In other words, there is a price for this sort of estimate, but we only have to pay one derivative.

As has already been mentioned, when considering (1.3), the following energy will be of interest,

$$E_k(\eta, \xi, \tau) = \frac{1}{2} \sum_{|\alpha|=k} \int_{T^d} [e^{2\eta} (D^\alpha \partial_\tau \xi)^2 + e^{2\eta - 2\tau} |\nabla D^\alpha \xi|^2] d\theta. \tag{5.3}$$

Lemma 5.2. *Let $\tau_0 \in \mathbb{R}_+$ and $I = [\tau_0, \infty)$. Assume η, $\xi \in C^\infty(\mathbb{R} \times T^d, \mathbb{R})$ and that*

$$0 < \gamma \leq \eta_\tau \leq 1 - \gamma < 1$$

on $I \times T^d$ where γ is a constant. Then

$$\frac{dE_k(\eta, \xi)}{d\tau} \leq -2\gamma E_k + \sum_{|\alpha|=k} \int_{T^d} f_\alpha(\eta, \xi) D^\alpha \partial_\tau \xi d\theta, \tag{5.4}$$

where

$$f_\alpha(\eta,\xi)=\partial_\tau(e^{2\eta}D^\alpha\partial_\tau\xi)-\nabla\cdot(e^{2\eta-2\tau}\nabla D^\alpha\xi). \tag{5.5}$$

If $\alpha=\hat{\alpha}+e_l$, where e_l is an element of $\mathbb{Z}^d$ whose l:th component is 1 and whose remaining components are zero, we have the following recursion formula for f_α:

$$f_\alpha=\partial_l f_{\hat{\alpha}}-2\partial_l\eta f_{\hat{\alpha}}-2\partial_l\partial_\tau\eta e^{2\eta}D^{\hat{\alpha}}\partial_\tau\xi+2\nabla(\partial_l\eta)\cdot(e^{2\eta-2\tau}\nabla D^{\hat{\alpha}}\xi). \tag{5.6}$$

Proof. Estimate

$$\frac{dE_k}{d\tau}=\sum_{|\alpha|=k}\int_{T^d}[\partial_\tau(e^{2\eta}D^\alpha\partial_\tau\xi)D^\alpha\partial_\tau\xi-\eta_\tau e^{2\eta}(D^\alpha\partial_\tau\xi)^2+$$

$$+(\eta_\tau-1)e^{2\eta-2\tau}|\nabla D^\alpha\xi|^2+e^{2\eta-2\tau}(\nabla D^\alpha\xi)\cdot(\nabla D^\alpha\partial_\tau\xi)]d\theta\leq$$

$$\leq -2\gamma E_k+\sum_{|\alpha|=k}\int_{T^d}[\partial_\tau(e^{2\eta}D^\alpha\partial_\tau\xi)-\nabla\cdot(e^{2\eta-2\tau}\nabla D^\alpha\xi)]D^\alpha\partial_\tau\xi d\theta,$$

and (5.4) follows. We have

$$f_\alpha=\partial_\tau(e^{2\eta}D^\alpha\partial_\tau\xi)-\nabla\cdot(e^{2\eta-2\tau}\nabla D^\alpha\xi)=$$

$$=\partial_\tau\partial_l(e^{2\eta}D^{\hat{\alpha}}\partial_\tau\xi)-\partial_\tau(2\partial_l\eta e^{2\eta}D^{\hat{\alpha}}\partial_\tau\xi)-\partial_l\nabla\cdot(e^{2\eta-2\tau}\nabla D^{\hat{\alpha}}\xi)+$$

$$+\nabla\cdot(2\partial_l\eta e^{2\eta-2\tau}\nabla D^{\hat{\alpha}}\xi)=\partial_l f_{\hat{\alpha}}-2\partial_l\eta f_{\hat{\alpha}}-2\partial_l\partial_\tau\eta e^{2\eta}D^{\hat{\alpha}}\partial_\tau\xi+$$

$$+2\nabla(\partial_l\eta)\cdot(e^{2\eta-2\tau}\nabla D^{\hat{\alpha}}\xi),$$

proving (5.6). □

6. Iteration

Consider the iteration (2.6)-(2.7). We will only be interested in the future evolution of solutions to the corresponding non-linear partial differential equation, and we will implicitly assume the time interval on which our estimates are valid to be $I=[\tau_0,\infty)$. Let

$$\tilde{P}_n=P_n-P_{n-1},\ \mathcal{E}_{n,k}=\mathcal{E}_k(P_n,\cdot),\ \tilde{\mathcal{E}}_{n,k}=\mathcal{E}_k(\tilde{P}_n,\cdot),$$

where $\mathcal{E}_k$ is defined by (2.3), and

$$\tilde{Q}_n=Q_n-Q_{n-1},\ E_{n,k}=E_k(P_{n-1},Q_n,\cdot),\ \tilde{E}_{n,k}=E_k(P_{n-1},\tilde{Q}_n,\cdot)$$

where E_k is defined in (5.3). Observe that these expressions are all independent of n if we evaluate them at τ_0, that $\mathcal{E}_{n,k}$ is defined if $n\geq 0$, $\tilde{\mathcal{E}}_{n,k}$ and $E_{n,k}$ are defined if $n\geq 1$ and $\tilde{E}_{n,k}$ is defined if $n\geq 2$.

Conditions and conventions concerning initial data. *We only consider initial data (p_0,p_1,q_0,q_1) with the property that there is a $\gamma>0$ such that*

$$0<2\gamma\leq p_1(\theta)\leq 1-2\gamma<1\ \ \forall\ \theta\in T^d. \tag{6.1}$$

Secondly, ϵ_k, ν_k and e_k will be taken to be constants satisfying

$$\mathcal{E}_{n,k}^{1/2}(\tau_0)\leq\epsilon_k,\ \mathcal{F}_k^{1/2}(P_n,\tau_0)\leq\nu_k,\ E_{n,k}^{1/2}(\tau_0)\leq e_k \tag{6.2}$$

for $k=0,...,m_d$, $m_d=2[d/2]+2$, with ϵ_k and e_k positive.

Induction hypothesis. *We assume that*

$$\mathcal{E}_{n,k}^{1/2}(\tau) \le \epsilon_k + 1 \ \forall \tau \in I \tag{6.3}$$

and that the following inequality is fulfilled on $I \times T^d$:

$$|P_{n,\tau}(\tau,\theta) - P_{n,\tau}(\tau_0,\theta)| \le \Delta\gamma. \tag{6.4}$$

Here $\Delta\gamma$ should be suitably small relative to γ, but not too small. We will below assume $\Delta\gamma = \gamma/4$ to hold.

Observe that if (6.4) holds for n and m, then

$$|P_{n,\tau}(\tau,\theta) - P_{m,\tau}(\tau,\theta)| \le 2\Delta\gamma. \tag{6.5}$$

Observe also that (6.4) implies that

$$0 < 2\gamma - \Delta\gamma \le P_{n,\tau} \le 1 - (2\gamma - \Delta\gamma) < 1 \tag{6.6}$$

on $I \times T^d$. In the course of the argument, we will give inequalities involving $\gamma, \epsilon_k, \nu_k$ and e_k introduced in (6.1) and (6.2) such that if they are fulfilled, the induction hypothesis is propagated. By imposing additional requirements, one can then prove

$$\tilde{\mathcal{E}}_{n+1,\max}^{1/2} \le \frac{1}{2}\tilde{\mathcal{E}}_{n,\max}^{1/2} \tag{6.7}$$

where

$$\tilde{\mathcal{E}}_{n,\max}^{1/2} = \sup_{\tau\in I} \tilde{\mathcal{E}}_{n,0}^{1/2} + ... + \sup_{\tau\in I} \tilde{\mathcal{E}}_{n,m_d}^{1/2}. \tag{6.8}$$

Thus one obtains convergence. Observe that it makes sense to speak of the suprema once we have proven that (6.3) is valid for all n. Let us note some consequences of the induction hypothesis.

Lemma 6.1. *Assume (6.3) is satisfied up to and including* $n-1$, $k = 0, ..., m_d$ *and that (6.2) holds. Then*

$$\sum_{|\alpha|=k} \|\nabla D^\alpha P_{n-1}\|_{L^2(T^d,\mathbb{R})} \le C[\nu_k + (\epsilon_{k+1} + 1)(\tau - \tau_0)] \tag{6.9}$$

$$\sum_{|\alpha|=k} \|\nabla D^\alpha \tilde{P}_{n-1}\|_{L^2(T^d,\mathbb{R})} \le C\tilde{\mathcal{E}}_{n-1,\max}^{1/2} \cdot (\tau - \tau_0) \tag{6.10}$$

on I, *for* $0 \le k \le m_d - 1$ *Finally, for* $n \ge 2$,

$$|\tilde{P}_{n-1}(\tau,\theta)| \le C\tilde{\mathcal{E}}_{n-1,\max}^{1/2} \cdot (\tau - \tau_0) \tag{6.11}$$

on I.

Proof. The estimates (6.9) and (6.10) follow from Lemma 5.1. In order to prove (6.11), we estimate

$$|P_{n-1}(\tau,\theta) - P_{n-2}(\tau,\theta)| = |\int_{\tau_0}^{\tau} \tilde{P}_{n-1,\tau}(s,\theta)ds| \le C\tilde{\mathcal{E}}_{n-1,\max}^{1/2}(\tau - \tau_0),$$

where we have used Lemma 5.1 in order to obtain the last inequality. □

Lemma 6.2. *If (6.2) holds, then (6.3) holds for* $n = 0$.

Proof. See (4.3). □

Lemma 6.3. *There are constants c_d such that if (6.2) and (6.3) are fulfilled for $n-1$ and*

$$(\epsilon_2 + 1)e^{-\tau_0} \leq c_1\gamma, \tag{6.12}$$

if $d = 1$, and

$$[\nu_{m_d/2+1} + (\epsilon_{m_d/2+2} + 1)]e^{-2\tau_0} \leq c_d\gamma \tag{6.13}$$

if $d \geq 2$, then

$$\int_{\tau_0}^{\infty} e^{-2\tau}|\Delta P_{n-1}(\tau, \cdot)|d\tau \leq \Delta\gamma/2$$

on T^d.

Remark. The expression $(\nu_2 + \epsilon_3)\exp(-2\tau_0)$ could also be used as the left hand side of (6.12) if one is prepared to keep track of one more derivative, c.f. the higher dimensional case of the argument presented in this paper.

Proof. We get a division into two cases depending on the dimension d. If $d \geq 2$, then $m_d \geq 4$, so that $m_d/2+1 \leq m_d - 1$. In consequence, we can use the estimates of Lemma 6.1 to obtain

$$\int_{\tau_0}^{\infty} e^{-2\tau}|\Delta P_{n-1}(\tau, \cdot)|d\tau \leq C\int_{\tau_0}^{\infty} e^{-2\tau}[\nu_{m_d/2+1} + (\epsilon_{m_d/2+2} + 1)(\tau - \tau_0)]d\tau \leq$$

$$\leq C[\nu_{m_d/2+1} + (\epsilon_{m_d/2+2} + 1)]e^{-2\tau_0}.$$

If $d = 1$, we do not have enough control to ensure the linear growth of the third spatial derivative in the L^2-norm, and therefore have to resort to using our control on $\mathcal{E}_{n-1,2}$. We get

$$\|P_{n-1,\theta\theta}\|_{C(S^1)} \leq C'\|P_{n-1,\theta\theta\theta}\|_{L^2(S^1)} \leq Ce^{\tau}(\epsilon_2 + 1),$$

where we have used (6.3). Thus

$$\int_{\tau_0}^{\infty} e^{-2\tau}|P_{n-1,\theta\theta}(\tau, \cdot)|d\tau \leq Ce^{-\tau_0}(\epsilon_2 + 1)$$

and the lemma follows. □

Lemma 6.4. *If the conditions of Lemma 6.3 and (6.2) hold, the inductive hypotheses (6.3) and (6.4) are satisfied for $n = 0$.*

Proof. The lemma follows by combining Lemma 6.2, 6.3 and (2.8). □

7. The n:th step

The first task is to estimate the behaviour of $E_{n,k}$. The point of the argument is to demand that all the iterates are such that $P_{n,\tau}$ belongs to a region where $Q_{n,\tau}$, at least intuitively, should decay to zero exponentially. Consequently, we hope to achieve an exponential decay for the energies $E_{n,k}$. This is in fact the case, but as the argument is constructed, the natural estimate that appears is a polynomial times an exponentially decaying factor. The polynomials that appear have an important property we wish to formalize.

Definition 7.1. Let $\mathcal{P}$ be a polynomial in $\tau - \tau_0$ depending on the ϵ_k, e_k and ν_k for $k = 0, ..., m_d$. We say that $\mathcal{P}$ is a *Q-dominated polynomial* if the coefficients of $\mathcal{P}$ are polynomial in e_k, ϵ_k and ν_k and go to zero when the e_k go to zero while the other expressions are fixed.

Remark. The Q-dominated polynomials we will consider will always be independent of n.

Lemma 7.1. *Assume that (6.2), (6.3) and (6.4) are satisfied for $n-1$ and let $0 \le k \le m_d$. Then*

$$E^{1/2}_{n,k}(\tau) \le \mathcal{P}_k(\tau - \tau_0)\exp[-\gamma(\tau - \tau_0)] \tag{7.1}$$

on I, where $\mathcal{P}_k$ is a Q-dominated polynomial. Furthermore,

$$\sum_{|\alpha|=k}\{\|e^{P_{n-1}}D^\alpha\partial_\tau Q_n\|_{C(T^d,\mathbb{R})} + \|e^{P_{n-1}-\tau}\nabla D^\alpha Q_n\|_{C(T^d,\mathbb{R})}\} \le \tag{7.2}$$

$$\le \mathcal{Q}_{k+1}(\tau - \tau_0)\exp[-\gamma(\tau - \tau_0)]$$

for $k = 0, ..., m_d/2$, where the $\mathcal{Q}_k$ are Q-dominated polynomials.

Let us make some preliminary observations. By Lemma 5.2, we have

$$\frac{dE_{n,k}}{d\tau} \le -2\gamma E_{n,k} + \sum_{|\alpha|=k}\int_{T^d} f_{n,\alpha}D^\alpha\partial_\tau Q_n d\theta \tag{7.3}$$

where

$$f_{n,\alpha} = \partial_\tau(e^{2P_{n-1}}D^\alpha\partial_\tau Q_n) - \nabla\cdot(e^{2P_{n-1}-2\tau}\nabla D^\alpha Q_n).$$

If $\alpha = \hat{\alpha} + e_l$, where e_l is an element of $\mathbb{Z}^d$ whose l:th component is 1 and whose remaining components are zero, we have the following recursion formula for $f_{n,\alpha}$:

$$f_{n,\alpha} = \partial_l f_{n,\hat{\alpha}} - 2\partial_l P_{n-1} f_{n,\hat{\alpha}} - 2\partial_l\partial_\tau P_{n-1}e^{2P_{n-1}}D^{\hat{\alpha}}\partial_\tau Q_n + \tag{7.4}$$

$$+2\nabla(\partial_l P_{n-1})\cdot(e^{2P_{n-1}-2\tau}\nabla D^{\hat{\alpha}}Q_n).$$

Lemma 7.2. *If*

$$\|e^{-P_{n-1}}f_{n,\alpha}\|_{L^2(T^d,\mathbb{R})} \le \pi_k\exp[-\gamma(\tau - \tau_0)] \tag{7.5}$$

for all α such that $|\alpha| = k$, where π_k is a Q-dominated polynomial independent of n, then an inequality of the form (7.1) holds.

Proof. By (7.3) and (7.5) we have

$$\frac{dE_{n,k}}{d\tau} \le -2\gamma E_{n,k} + \sum_{|\alpha|=k}\int_{T^d} f_{n,\alpha}D^\alpha\partial_\tau Q_n d\theta \le$$

$$\le -2\gamma E_{n,k} + \sum_{|\alpha|=k}\|e^{-P_{n-1}}f_{n,\alpha}\|_{L^2(T^d,\mathbb{R})}\|e^{P_{n-1}}D^\alpha\partial_\tau Q_n\|_{L^2(T^d,\mathbb{R})} \le$$

$$\le -2\gamma E_{n,k} + \sqrt{2}C\pi_k\exp[-\gamma(\tau - \tau_0)]E^{1/2}_{n,k}.$$

Thus

$$\frac{d}{d\tau}\{\exp[2\gamma(\tau - \tau_0)]E_{n,k}\} \le \sqrt{2}C\pi_k\{\exp[2\gamma(\tau - \tau_0)]E_{n,k}\}^{1/2}$$

which can be integrated to

$$\{\exp[2\gamma(\tau-\tau_0)]E_{n,k}\}^{1/2} \le E_{n,k}^{1/2}(\tau_0) + \int_{\tau_0}^{\tau} 2^{-1/2}C\pi_k(s-\tau_0)ds,$$

and the lemma follows. □

Proof of Lemma 7.1. Observe that

$$f_{n,0} = \partial_\tau(e^{2P_{n-1}}\partial_\tau Q_n) - \nabla\cdot(e^{2P_{n-1}-2\tau}\nabla Q_n) =$$

$$= e^{2P_{n-1}}[Q_{n,\tau\tau} + 2P_{n-1,\tau}Q_{n,\tau} - e^{-2\tau}\Delta Q_n - 2e^{-2\tau}\nabla P_{n-1}\cdot\nabla Q_n] = 0.$$

Let us also observe that when the two first terms in (7.4) hit an expression involving $\exp(2P_{n-1})$, then the effect is to differentiate the expression regarding the exponential expression mentioned as a constant. Inductively we thus get the conclusion that if $|\alpha| = k+1$, then $f_{n,\alpha}$ consists of terms of the form

$$C_{\alpha_1,\alpha_2}e^{2P_{n-1}}D^{\alpha_1}\partial_\tau P_{n-1}D^{\alpha_2}\partial_\tau Q_n \tag{7.6}$$

and

$$B_{\alpha_1,\alpha_2}e^{2P_{n-1}-2\tau}\nabla D^{\alpha_1}P_{n-1}\cdot\nabla D^{\alpha_2}Q_n \tag{7.7}$$

where $|\alpha_1| \ge 1$ and $\alpha_1+\alpha_2 = \alpha$. We will have to use different estimates for different k:s.

Zeroth order energy. Observe that

$$\frac{dE_{n,0}}{d\tau} \le -2\gamma E_{n,0}$$

so that

$$E_{n,0}^{1/2}(\tau) \le e_0\exp[-\gamma(\tau-\tau_0)]$$

for $\tau \ge \tau_0$. Thus (7.1) holds for $k=0$, with $\mathcal{P}_0 = e_0$.

Intermediate order energies. The condition $1 \le k \le m_d/2$ defines what we mean by intermediate energies. We carry out an argument by induction. Assume that we have (7.1) up to and including k, $0 \le k \le m_d/2-1$. By Lemma 7.2, we need to consider

$$\|e^{-P_{n-1}}f_{n,\alpha}\|_{L^2(T^d,\mathbb{R})}$$

for $|\alpha| = k+1$. In order to deal with terms of the form (7.6), we need to estimate

$$\|e^{P_{n-1}}D^{\alpha_1}\partial_\tau P_{n-1}D^{\alpha_2}\partial_\tau Q_n\|_{L^2(T^d,\mathbb{R})} \tag{7.8}$$

where $\alpha_1+\alpha_2 = \alpha$ and $|\alpha_1| \ge 1$. By Lemma 5.1 and the induction hypothesis on n (6.3) we can take out $D^{\alpha_1}\partial_\tau P_{n-1}$ in the sup norm. The remaining part is bounded by $\sqrt{2}E_{n,|\alpha_2|}^{1/2}$. By the induction hypothesis on k and the fact that $|\alpha_2| \le k$, we get the conclusion that

$$\|e^{P_{n-1}}D^{\alpha_1}\partial_\tau P_{n-1}D^{\alpha_2}\partial_\tau Q_n\|_{L^2(T^d,\mathbb{R})} \le \pi_{k+1}\exp[-\gamma(\tau-\tau_0)].$$

The estimate for terms of the form (7.7) is similar, and we can thus apply Lemma 7.2 in order to obtain (7.1) for the intermediate energies.

High order energies. By high order we mean $m_d/2+1 \leq k \leq m_d$. Since we cannot assume to control the sup norm of the derivatives of P_{n-1} indefinitely, we need to change our method. Observe that if $|\alpha| \leq k \leq m_d/2$, then

$$\|D^\alpha(e^{P_{n-1}}Q_{n,\tau})\|_{L^2(T^d,\mathbb{R})} \leq \pi_\alpha \sum_{j=0}^{k} E_{n,j}^{1/2} \tag{7.9}$$

for some polynomial π_α, c.f. Lemma 6.1. The argument for $\exp(P_{n-1}-\tau)\nabla Q_n$ is similar, and we get

$$\|e^{P_{n-1}}Q_{n,\tau}\|_{C(T^d,\mathbb{R})} + \|e^{P_{n-1}-\tau}\nabla Q_n\|_{C(T^d,\mathbb{R})} \leq \mathcal{Q}_1 \exp[-\gamma(\tau-\tau_0)] \tag{7.10}$$

by Sobolev embedding, where $\mathcal{Q}_1$ is a Q-dominated polynomial. Let us assume that (7.1) is satisfied up to and including $m_d/2+l \leq m_d-1$, and that (7.2) is satisfied for $0 \leq k \leq l$. For $l=0$ we know this to be true. We wish to prove an estimate of the form (7.5) and as for the intermediate energies, we need to consider (7.8) where $\alpha_1+\alpha_2=\alpha$, $|\alpha| = m_d/2+l+1$ and $|\alpha_1| \geq 1$. If $|\alpha_1| \leq m_d/2$, we can take out $D^{\alpha_1}\partial_\tau P_{n-1}$ in the sup norm to obtain

$$\|e^{P_{n-1}}D^{\alpha_1}\partial_\tau P_{n-1}D^{\alpha_2}\partial_\tau Q_n\|_{L^2(T^d,\mathbb{R})} \leq \sqrt{2}\|D^{\alpha_1}\partial_\tau P_{n-1}\|_{C(T^d,\mathbb{R})}E_{n,|\alpha_2|}^{1/2}$$

which satisfies a bound as in (7.5) by Lemma 5.1, the induction hypotheses on l and the fact that $|\alpha_2| \leq m_d/2+l$. If $|\alpha_2| \leq l$ we can take out $e^{P_{n-1}}D^{\alpha_2}\partial_\tau Q_n$ in the sup norm in order to achieve a similar bound using the induction hypothesis on l and the boundedness of $\mathcal{E}_{n,|\alpha_1|}$. The argument for terms of the form (7.7) is similar. Since $|\alpha_1| > m_d/2$ and $|\alpha_2| > l$ cannot occur at the same time, we are done. We have thus proven (7.1) for $k = m_d/2+l+1$. We now need to prove that (7.2) holds for $k=l+1$. However, this can be proven in the same way as (7.9); replace $Q_{n,\tau}$ in that inequality with $D^\alpha\partial_\tau Q_n$ for $|\alpha| \leq l+1$. □

Let us now turn to the problem of estimating $\mathcal{E}_{n,k}$.

Lemma 7.3. *Assume (6.2), (6.3) and (6.4) are fulfilled for $n-1$ and let $k = 0,...,m_d$. Then*

$$\mathcal{E}_{n,k}^{1/2}(\tau) \leq \mathcal{E}_{n,k}^{1/2}(\tau_0) + \int_{\tau_0}^{\tau} \mathcal{V}_k(s-\tau_0)\exp[-2\gamma(s-\tau_0)]ds, \tag{7.11}$$

where $\mathcal{V}_k$ is a Q-dominated polynomial.

Proof. Observe that

$$\frac{d\mathcal{E}_{n,k}}{d\tau} \leq \sqrt{2}\sum_{|\alpha|=k}\|D^\alpha[e^{2P_{n-1}}(Q_{n,\tau}^2 - e^{-2\tau}|\nabla Q_n|^2)]\|_{L^2(T^d,\mathbb{R})}\mathcal{E}_{n,k}^{1/2}.$$

It is thus of interest to estimate

$$D^\alpha(e^{2P_{n-1}}Q_{n,\tau}^2) =$$

$$= \sum_{j=0}^{k}\sum_{\beta_1+...+\beta_{j+2}=\alpha} C_{\beta_1,...,\beta_{j+2}}D^{\beta_1}P_{n-1}\cdots D^{\beta_j}P_{n-1}e^{2P_{n-1}}D^{\beta_{j+1}}\partial_\tau Q_n D^{\beta_{j+2}}\partial_\tau Q_n$$

in L^2-norm. Consider a term in the sum. Observe that at most one $|\beta_l|$ can be greater than $m_d/2$. Combining Lemma 6.1 and 7.1, we conclude that

$$\|D^\alpha(e^{2P_{n-1}}Q_{n,\tau}^2)\|_{L^2(T^d,\mathbb{R})} \leq \pi_k \exp[-2\gamma(\tau-\tau_0)]$$

where π_k is a Q-dominated polynomial. The argument for

$$\|D^\alpha(e^{2P_{n-1}-2\tau}|\nabla Q_n|^2)\|_{L^2(T^d,\mathbb{R})}$$

is similar, and we obtain

$$\frac{d\mathcal{E}_{n,k}}{d\tau} \leq 2\mathcal{V}_k \exp[-2\gamma(\tau-\tau_0)]\mathcal{E}_{n,k}^{1/2},$$

which can be integrated to (7.11). □

We now wish to specify conditions that imply (6.3) and (6.4) for n.

Lemma 7.4. *There are constants c_k, and non-negative integers i_k and j_k such that if*

$$e_k \leq c_k(1+\epsilon_0+\epsilon_{m_d}+\nu_{m_d})^{-i_k}\gamma^{j_k} \tag{7.12}$$

for $k=0,...,m_d$, and the relevant condition in Lemma 6.3 is satisfied, then (6.3) and (6.4) hold for n if they hold for $n-1$.

Proof. Consider (7.11) in order to prove that (6.3) holds for n. We have

$$\mathcal{V}_k(s-\tau_0) = \sum_{j=1}^{l_k} a_{k,j}(s-\tau_0)^j$$

where the $a_{k,j}$ are polynomials in ϵ_i, e_i and ν_i $i=0,...,m_d$ (observe that we can consider the coefficients to be polynomials in ν_{m_d}, ϵ_0, ϵ_{m_d} and the e_i if we wish). We thus get

$$\int_{\tau_0}^{\tau} \mathcal{V}_k(s-\tau_0)\exp[-2\gamma(s-\tau_0)]ds = \sum_{j=1}^{l_k}\int_{\tau_0}^{\tau} a_{k,j}(s-\tau_0)^j \exp[-2\gamma(s-\tau_0)]ds \leq$$

$$\leq \sum_{j=1}^{l_k} a_{k,j}\frac{j!}{2^{j+1}\gamma^{j+1}}.$$

We wish this expression to be less than or equal to 1, and since $a_{k,j}$ is polynomial in e_i, ϵ_i and ν_i with each term containing at least one factor e_i, we conclude that (6.3) holds assuming that an inequality of the form (7.12) holds. By (2.6) and Lemma 6.3, we have

$$|P_{n,\tau}(\tau,\theta)-P_{n,\tau}(\tau_0,\theta)| \leq \int_{\tau_0}^{\tau}[e^{-2s}|\Delta P_n| + e^{2P_{n-1}}(Q_{n,\tau}^2 + e^{-2s}|\nabla Q_n|^2)]ds \leq$$

$$\leq \Delta\gamma/2 + 2\int_{\tau_0}^{\tau}\mathcal{Q}_1^2\exp[-2\gamma(s-\tau_0)]ds,$$

by Lemma 6.3. The remaining statement of the lemma follows. □

8. Convergence

Consider the difference between (2.7) for $n+1$ and n. We have

$$\tilde{Q}_{n+1,\tau\tau} - e^{-2\tau}\Delta\tilde{Q}_{n+1} + 2(P_{n,\tau}\tilde{Q}_{n+1,\tau} - e^{-2\tau}\nabla P_n\cdot\nabla\tilde{Q}_{n+1}) =$$

$$= -2(\tilde{P}_{n,\tau}Q_{n,\tau} - e^{-2\tau}\nabla\tilde{P}_n\cdot\nabla Q_n)$$

$$\tilde{Q}_{n+1}(\tau_0,\cdot) = 0 \qquad (8.1)$$
$$\tilde{Q}_{n+1,\tau}(\tau_0,\cdot) = 0.$$

Since (6.4) is fulfilled for all $n \geq 0$ and (6.1) holds, Lemma 5.2 yields

$$\frac{d\tilde{E}_{n,k}}{d\tau} \leq -2\gamma\tilde{E}_{n,k} + \sum_{|\alpha|=k}\int_{T^d}\tilde{f}_{n,\alpha}D^\alpha\partial_\tau\tilde{Q}_n d\theta, \qquad (8.2)$$

where

$$\tilde{f}_{n,\alpha} = \partial_\tau(e^{2P_{n-1}}D^\alpha\partial_\tau\tilde{Q}_n) - \nabla\cdot(e^{2P_{n-1}-2\tau}\nabla D^\alpha\tilde{Q}_n).$$

We have

(8.3)
$$\tilde{f}_{n+1,0} = e^{2P_n}[\tilde{Q}_{n+1,\tau\tau} - e^{-2\tau}\Delta\tilde{Q}_{n+1} + 2(P_{n,\tau}\tilde{Q}_{n+1,\tau} - e^{-2\tau}\nabla P_n\cdot\nabla\tilde{Q}_{n+1})] =$$
$$= -2e^{2P_n}(\tilde{P}_{n,\tau}Q_{n,\tau} - e^{-2\tau}\nabla\tilde{P}_n\cdot\nabla Q_n).$$

The recursion formula is the same as before: if we have $\alpha = \hat{\alpha} + e_l$, then

$$\tilde{f}_{n+1,\alpha} = \partial_l\tilde{f}_{n+1,\hat{\alpha}} - 2\partial_l P_n\tilde{f}_{n+1,\hat{\alpha}} -$$
$$-2\partial_l\partial_\tau P_n e^{2P_n}D^{\hat{\alpha}}\partial_\tau\tilde{Q}_{n+1} + 2e^{2P_n-2\tau}\nabla\partial_l P_n\cdot\nabla D^{\hat{\alpha}}\tilde{Q}_{n+1}.$$

Lemma 8.1. *Assume that conditions as in Lemma 7.4 are fulfilled, so that (6.3)-(6.5) are fulfilled for all $n \geq 0$. Let $0 \leq k \leq m_d$, then*

$$\tilde{E}^{1/2}_{n+1,k} \leq \tilde{\mathcal{E}}^{1/2}_{n,\max}\tilde{\mathcal{P}}_k(\tau-\tau_0)\exp[-(\gamma-2\Delta\gamma)(\tau-\tau_0)], \qquad (8.4)$$

where $\tilde{\mathcal{P}}_k$ is a Q-dominated polynomial and $\tilde{\mathcal{E}}^{1/2}_{n,\max}$ is given by (6.8). Furthermore

$$\sum_{|\alpha|=k}\{\|e^{P_n}D^\alpha\partial_\tau\tilde{Q}_{n+1}\|_{C(T^d,\mathbb{R})} + \|e^{P_n-\tau}\nabla D^\alpha\tilde{Q}_{n+1}\|_{C(T^d,\mathbb{R})}\} \leq \qquad (8.5)$$
$$\leq \tilde{\mathcal{E}}^{1/2}_{n,\max}\tilde{\mathcal{Q}}_{k+1}(\tau-\tau_0)\exp[-(\gamma-2\Delta\gamma)(\tau-\tau_0)]$$

on I, for $k = 0,...,m_d/2$, where $\tilde{\mathcal{Q}}_k$ is a Q-dominated polynomial.

The proof of this statement is similar to the proof of Lemma 7.1. The proof of the following lemma is analogous to the proof of Lemma 7.2

Lemma 8.2. *If*

$$\|e^{-P_n}\tilde{f}_{n+1,\alpha}\|_{L^2(T^d,\mathbb{R})} \leq \tilde{\mathcal{E}}^{1/2}_{n,\max}\pi_k\exp[-(\gamma-2\Delta\gamma)(\tau-\tau_0)] \qquad (8.6)$$

for all α such that $|\alpha| = k$, where π_k is a Q-dominated polynomial independent of n, then an inequality of the form (8.4) holds.

Proof of Lemma 8.1. Observe that for $|\alpha| \geq 1$,

(8.7)
$$\tilde{f}_{n+1,\alpha} = \sum_{\alpha_1+\alpha_2=\alpha}A_{\alpha_1,\alpha_2}e^{2P_n}(D^{\alpha_1}\partial_\tau\tilde{P}_nD^{\alpha_2}\partial_\tau Q_n - e^{-2\tau}\nabla D^{\alpha_1}\tilde{P}_n\cdot\nabla D^{\alpha_2}Q_n) +$$
$$+ \sum_{\beta_1+\beta_2=\alpha,|\beta_1|\geq 1}B_{\beta_1,\beta_2}e^{2P_n}(D^{\beta_1}\partial_\tau P_nD^{\beta_2}\partial_\tau\tilde{Q}_{n+1} - e^{-2\tau}\nabla D^{\beta_1}P_n\cdot\nabla D^{\beta_2}\tilde{Q}_{n+1}).$$

In order to be able to apply Lemma 8.2, we need to estimate the left hand side of (8.6). By the above observation (8.7), it is enough to estimate the expressions

$$\|e^{P_n}D^{\alpha_1}\partial_\tau\tilde{P}_nD^{\alpha_2}\partial_\tau Q_n\|_{L^2(T^d,\mathbb{R})}+\|e^{P_n-2\tau}\nabla D^{\alpha_1}\tilde{P}_n\cdot\nabla D^{\alpha_2}Q_n\|_{L^2(T^d,\mathbb{R})} \tag{8.8}$$

and

$$\|e^{P_n}D^{\beta_1}\partial_\tau P_nD^{\beta_2}\partial_\tau\tilde{Q}_{n+1}\|_{L^2(T^d,\mathbb{R})}+\|e^{P_n-2\tau}\nabla D^{\beta_1}P_n\cdot\nabla D^{\beta_2}\tilde{Q}_{n+1}\|_{L^2(T^d,\mathbb{R})}. \tag{8.9}$$

Concerning expressions of the form (8.8), we need only apply Lemma 5.1, Lemma 7.1, (6.5) and the fact that one of $|\alpha_1|$ and $|\alpha_2|$ has to be less than or equal to $m_d/2$ in order to bound (8.8) by the right hand side of (8.6). Let us illustrate on the first term in (8.8) under the assumption that $|\alpha_1|\le m_d/2$. We have

$$\|e^{P_n}D^{\alpha_1}\partial_\tau\tilde{P}_nD^{\alpha_2}\partial_\tau Q_n\|_{L^2(T^d,\mathbb{R})}\le$$

$$\le\|e^{P_n-P_{n-1}}\|_{C(T^d,\mathbb{R})}\|D^{\alpha_1}\partial_\tau\tilde{P}_n\|_{C(T^d,\mathbb{R})}\|e^{P_{n-1}}D^{\alpha_2}\partial_\tau Q_n\|_{L^2(T^d,\mathbb{R})}\le$$

$$\le C\exp[2\Delta\gamma(\tau-\tau_0)]\tilde{\mathcal{E}}^{1/2}_{n,\max}\mathcal{P}_{|\alpha_2|}\exp[-\gamma(\tau-\tau_0)].$$

If $\alpha=0$, then only terms of the form (8.8) are of relevance, so in that case, (8.4) follows.

Intermediate energies. Let us now prove (8.4) for $0\le|\alpha|\le m_d/2$ by induction on $|\alpha|$. All we need to do is to prove that (8.9) is bounded by an expression as in the right hand side of (8.6). Assume (8.4) holds up to and including $k\le m_d/2-1$. Since $k+1\le m_d/2$ and $|\beta_1|\le|\alpha|=k+1$, we can use Lemma 5.1 in order to extract $D^{\beta_1}\partial_\tau P_n$ and $e^{-\tau}\nabla D^{\beta_1}P_n$ in the sup norm. Since $|\beta_2|\le|\alpha|-1\le k$, we can estimate what remains using (8.4) and the inductive hypothesis. By applying Lemma 8.2 we have thus proven (8.4) for $k\le m_d/2$.

High energies. The argument now proceeds as in the proof of Lemma 7.1. By Lemma 6.1, $D^\alpha P_n$ is bounded in the sup norm by a polynomial if $|\alpha|\le m_d/2$, so that the result concerning the intermediate energies yields the conclusion that

$$\|D^\alpha(e^{P_n}\tilde{Q}_{n+1,\tau})\|_{L^2(T^d,\mathbb{R})}$$

can be bounded by the right hand side of (8.5) if $|\alpha|\le m_d/2$. A similar statement holds for $e^{P_n-\tau}\nabla\tilde{Q}_{n+1}$, and (8.5) follows for $k=0$ by Sobolev embedding. Assume now that (8.4) holds up to and including $k=m_d/2+l\le m_d-1$ and that (8.5) holds for k up to and including l. For $l=0$, we know this to be true. Consider now (8.9) with $\beta_1+\beta_2=\alpha$, $|\beta_1|\ge1$ and $|\alpha|=k+1=m_d/2+l+1$. If $|\beta_1|\le m_d/2$, we can take out $D^{\beta_1}\partial_\tau P_n$ and $e^{-\tau}\nabla D^{\beta_1}P_n$ in the sup norm by Lemma 5.1 and then apply the induction hypothesis to what remains since $|\beta_2|\le|\alpha|-1\le m_d/2+l$. If $|\beta_2|\le l$, we can take out $e^{P_n}D^{\beta_2}\partial_\tau\tilde{Q}_{n+1}$ and $e^{P_n-\tau}\nabla D^{\beta_2}\tilde{Q}_{n+1}$ in the sup norm, using the inductive hypothesis. What remains is bounded by $\mathcal{E}^{1/2}_{n,|\beta_1|}$ and (8.4) follows for $m_d/2+l+1$ since one of the inequalities $|\beta_1|\le m_d/2$ and $|\beta_2|\le l$ must hold. In order to prove (8.5) for $l+1$, we proceed by Sobolev embedding as before. □

Finally, let us consider $\tilde{\mathcal{E}}_{n,k}$.

Lemma 8.3. *Assume that conditions as in Lemma 7.4 are fulfilled, so that (6.3)-(6.5) are fulfilled for all $n \geq 0$. We have*

$$\tilde{\mathcal{E}}^{1/2}_{n+1,k} \leq \tilde{\mathcal{E}}^{1/2}_{n,\max} \int_{\tau_0}^{\tau} \tilde{\mathcal{V}}_k(s-\tau_0)\exp[-2(\gamma - 2\Delta\gamma)(s-\tau_0)]ds, \tag{8.10}$$

on I, for $k = 0, ..., m_d$, where $\tilde{\mathcal{V}}_k$ is a Q-dominated polynomial.

We need some preliminaries. Consider the difference of (2.6) for $n+1$ and n. We have

$$\tilde{P}_{n+1,\tau\tau} - e^{-2\tau}\Delta\tilde{P}_{n+1} = e^{2P_n}(Q^2_{n+1,\tau} - e^{-2\tau}|\nabla Q_{n+1}|^2) -$$

$$-e^{2P_{n-1}}(Q^2_{n,\tau} - e^{-2\tau}|\nabla Q_n|^2) = e^{2P_n}(Q_{n+1,\tau}\tilde{Q}_{n+1,\tau} - e^{-2\tau}\nabla Q_{n+1}\cdot\nabla\tilde{Q}_{n+1}) +$$

$$+e^{2P_n}(Q_{n,\tau}\tilde{Q}_{n+1,\tau} - e^{-2\tau}\nabla Q_n\cdot\nabla\tilde{Q}_{n+1}) + (e^{2P_n} - e^{2P_{n-1}})(Q^2_{n,\tau} - e^{-2\tau}|\nabla Q_n|^2) =$$

$$= g_1 + g_2 + g_3.$$

Estimate

$$\frac{d\tilde{\mathcal{E}}_{n+1,k}}{d\tau} \leq \sum_{|\alpha|=k}\int_{T^d} D^\alpha(g_1+g_2+g_3)D^\alpha\partial_\tau\tilde{P}_{n+1}d\theta. \tag{8.11}$$

Lemma 8.4. *If*

$$\|D^\alpha(g_1+g_2+g_3)\|_{L^2(T^d,\mathbb{R})} \leq \tilde{\mathcal{E}}^{1/2}_{n,\max}\pi_{|\alpha|}\exp[-2(\gamma-2\Delta\gamma)(\tau-\tau_0)] \tag{8.12}$$

for $|\alpha| \leq m_d$, where $\pi_{|\alpha|}$ is a Q-dominated polynomial, then (8.10) follows.

Proof. By (8.11),

$$\frac{d\tilde{\mathcal{E}}_{n+1,k}}{d\tau} \leq \sum_{|\alpha|=k}\sqrt{2}\|D^\alpha(g_1+g_2+g_3)\|_{L^2(T^d,\mathbb{R})}\tilde{\mathcal{E}}^{1/2}_{n+1,k}$$

which can be integrated to (8.10) given the assumptions of the lemma. □

Proof of Lemma 8.3. Consider the contribution of the first term in g_1 to $D^\alpha g_1$. If more than $m_d/2$ derivatives hit one of P_n, $Q_{n+1,\tau}$ or $\tilde{Q}_{n+1,\tau}$, then everything else can be taken out in the sup norm, and we obtain an estimate of the form (8.12). In fact the estimate is a bit better, but we will need the form (8.12) for the other terms. The argument for the contribution of the second term in g_1 is similar. The expression $D^\alpha g_2$ can be controlled by similar arguments, but we loose one factor $\exp[-2\Delta\gamma(\tau-\tau_0)]$ in decay due to the fact that we have to compensate that we in some situations have e^{P_n} were we would prefer to have $e^{P_{n-1}}$.

Consider

$$D^\alpha[(e^{2P_n} - e^{2P_{n-1}})Q^2_{n,\tau}].$$

This expression is a linear combination of the following types of terms

$$(e^{2P_n}D^{\alpha_1}P_n\cdots D^{\alpha_l}P_n - e^{2P_{n-1}}D^{\alpha_1}P_{n-1}\cdots D^{\alpha_l}P_{n-1})D^{\alpha_{l+1}}\partial_\tau Q_n D^{\alpha_{l+2}}\partial_\tau Q_n$$

where $\alpha_1 + ... + \alpha_{l+2} = \alpha$. These terms can in turn be written as a linear combination of terms such as

$$e^{2P_{n-1}}D^{\alpha_1}P_n\cdots D^{\alpha_r}\tilde{P}_n\cdots D^{\alpha_l}P_{n-1}D^{\alpha_{l+1}}\partial_\tau Q_n D^{\alpha_{l+2}}\partial_\tau Q_n \tag{8.13}$$

and

$$(e^{2P_n} - e^{2P_{n-1}})D^{\alpha_1}P_n \cdots D^{\alpha_l}P_n D^{\alpha_{l+1}}\partial_\tau Q_n D^{\alpha_{l+2}}\partial_\tau Q_n. \tag{8.14}$$

Terms of the form (8.13) can be handled using Lemma 6.1 and 7.1, since at most one $|\alpha_j|$ can be bigger than $m_d/2$. For terms of type (8.14), we estimate

$$|(e^{2P_n} - e^{2P_{n-1}})D^{\alpha_1}P_n \cdots D^{\alpha_l}P_n D^{\alpha_{l+1}}\partial_\tau Q_n D^{\alpha_{l+2}}\partial_\tau Q_n| \leq$$

$$\leq 2|P_n - P_{n-1}| \exp[4\Delta\gamma(\tau-\tau_0)]e^{2P_{n-1}}|D^{\alpha_1}P_n \cdots D^{\alpha_l}P_n D^{\alpha_{l+1}}\partial_\tau Q_n D^{\alpha_{l+2}}\partial_\tau Q_n|$$

which we can deal with, using (6.11). The contribution from the second term in g_3 can be estimated similarly. □

9. Conclusions

Theorem 9.1. *Let $0 < \gamma \leq 1/4$, $\tau_0 \geq 0$ and define $m_d = 2[d/2] + 2$. There are constants $c_{i,d}$, $i = 1, 2$, and integers $l_{i,d}$, $i = 1, 2$ depending on the dimension such that the following holds. If $d = 1$, let ϵ_2 and τ_0 be such that*

$$(1 + \epsilon_2)e^{-\tau_0} \leq c_{1,1}\gamma. \tag{9.1}$$

If $d > 1$, let $\nu_{m_d/2+1}$, $\epsilon_{m_d/2+2}$ and τ_0 be such that

$$[\nu_{m_d/2+1} + \epsilon_{m_d/2+2} + 1]e^{-2\tau_0} \leq c_{1,d}\gamma. \tag{9.2}$$

Specify the remaining ϵ_k and ν_k, $k = 0, ..., m_d$ freely. Assume furthermore that

$$e_k \leq c_{2,d}(1 + \epsilon_0 + \epsilon_{m_d} + \nu_{m_d})^{-l_{1,d}}\gamma^{l_{2,d}} \tag{9.3}$$

for $k = 0, ..., m_d$. Then every quadruple of functions (p_0, p_1, q_0, q_1) satisfying

$$2\gamma \leq p_1 \leq 1 - 2\gamma \tag{9.4}$$

and (6.2), yield upon solving (2.1) and (2.2) smooth solutions on $[\tau_0, \infty)$ with the following properties. For all non-negative integers k, there are polynomials $\Xi_{i,k}$, $i = 1, ..., 6$ in $\tau - \tau_0$, and $v, w, q, r \in C^\infty(T^d, \mathbb{R})$, where

$$0 < \gamma \leq v \leq 1 - \gamma < 1 \tag{9.5}$$

on T^d such that

$$\|P_\tau - v\|_{C^k(T^d,\mathbb{R})} \leq \Xi_{1,k}\exp[-2\gamma(\tau - \tau_0)], \tag{9.6}$$

$$\|P - \rho\|_{C^k(T^d,\mathbb{R})} \leq \Xi_{2,k}\exp[-2\gamma(\tau - \tau_0)], \tag{9.7}$$

$$\sum_{|\alpha|\leq k} \|e^{2\rho}D^\alpha\partial_\tau Q\|_{C(T^d,\mathbb{R})} \leq \Xi_{3,k}, \tag{9.8}$$

$$\sum_{|\alpha|\leq k} \|e^{2\rho}[D^\alpha Q - D^\alpha q]\|_{C(T^d,\mathbb{R})} \leq \Xi_{4,k}, \tag{9.9}$$

$$\|e^{2\rho}Q_\tau - r\|_{C^k(T^d,\mathbb{R})} \leq \Xi_{5,k}\exp[-2\gamma(\tau - \tau_0)], \tag{9.10}$$

and

$$\|e^{2\rho}(Q - q) + \frac{r}{2v}\|_{C^k(T^d,\mathbb{R})} \leq \Xi_{6,k}\exp[-2\gamma(\tau - \tau_0)] \tag{9.11}$$

for all $\tau \in [\tau_0, \infty)$, where $\rho = v \cdot (\tau - \tau_0) + w$. Finally, assume $v \in C^\infty(T^d, \mathbb{R})$ satisfies (9.5), and that

$$\sum_{|\alpha| \leq m_d + 1} \|e^P D^\alpha \partial_\tau Q\|_{C(T^d,\mathbb{R})} + \|P_\tau - v\|_{C^{m_d+1}(T^d,\mathbb{R})} \leq C e^{-\epsilon\tau} \tag{9.12}$$

where $\epsilon > 0$ and $P, Q \in C^\infty(\mathbb{R} \times T^d, \mathbb{R})$. Let

$$e'_k(\tau) = E_k^{1/2}(P, Q, \tau),\ \epsilon'_k(\tau) = \mathcal{E}_k^{1/2}(P, \tau),\ \nu'_k(\tau) = \mathcal{F}_k^{1/2}(P, \tau)$$

and $\gamma' = \gamma/4$. Then for τ'_0 big enough, (9.1)-(9.4) will be satisfied with e_k replaced by $e'_k(\tau'_0)$, τ_0 replaced with τ'_0, γ replaced with γ' etc.

Remark. The last part of the theorem is intended to illustrate the fact that there is in some sense an equivalence between the conditions (9.1)-(9.4) and the asymptotics (9.6)-(9.11).

Let us briefly comment on the conditions before we turn to the proof. The idea of the argument is to see to it that P_τ is always bounded away from 0 below and 1 above. As can be seen by Lemma 6.3, the conditions (9.1) and (9.2) are there to ensure that the term $e^{-2\tau}\Delta P$ in (2.1) does not push P_τ out of this interval. The condition (9.3) is then there to ensure that the remaining terms in (2.1) do not push us away from the desired region.

Proof. Observe first that (6.3) and (6.4) hold for $n = 0$ due to Lemma 6.4 and the assumption that (9.1) or (9.2) hold. By Lemma 7.4, an estimate of the form (9.3), together with estimates of the form (9.1) or (9.2) imply that (6.3) and (6.4) hold for all $n \geq 0$. By (8.10), an argument similar to the proof of Lemma 7.4 yields that a condition of the type (9.3) implies

$$\tilde{\mathcal{E}}_{n+1,\max}^{1/2} \leq \frac{1}{2} \tilde{\mathcal{E}}_{n,\max}^{1/2}. \tag{9.13}$$

Let $T \in I$ and let us consider the convergence on $[\tau_0, T] \times T^d$. By (6.11), we conclude that P_n is a Cauchy sequence in sup norm. Observe that under these circumstances, factors of the type e^{2P_n} and $e^{-2\tau}$ are of no importance, since we are considering a finite time interval, and since the sequence P_n is uniformly bounded on this finite time interval. By (8.5), (9.13) and the equations, we conclude that

$$D^\alpha \partial_\tau^l P_n \quad \text{and} \quad D^\alpha \partial_\tau^l Q_n$$

are Cauchy sequences in $C([\tau_0, T] \times T^d, \mathbb{R})$ for $1 \leq |\alpha| + l \leq m_d/2 + 1$. The convergence of Q_n follows from the convergence of $Q_{n,\tau}$, the finiteness of the time interval and the fact that the iterates coincide at τ_0. In particular, the iteration yields C^2 solutions to the equations for $\tau \in [\tau_0, \infty)$. By Proposition 3.1, the solutions will be smooth if the initial values consist of smooth functions (the transformation $t = -e^{-\tau}$ yields an equation of the right form). Furthermore, if we let

$$E_k(\tau) = E_k(P, Q, \tau)$$

and

$$\mathcal{E}_k(\tau) = \mathcal{E}_k(P, \tau),$$

then these expressions will satisfy the estimates (7.1), (7.2) and (6.3).

Lemma 9.1. *There are polynomials $\mathcal{P}_k$ and $\mathcal{Q}_k$ and constants c_k such that*

$$E_k^{1/2}(\tau) \leq \mathcal{P}_k(\tau-\tau_0)\exp[-\gamma(\tau-\tau_0)], \tag{9.14}$$

$$\sum_{|\alpha|=k}\{\|e^P D^\alpha \partial_\tau Q\|_{C(T^d,\mathbb{R})} + \|e^{P-\tau}\nabla D^\alpha Q\|_{C(T^d,\mathbb{R})}\} \leq \tag{9.15}$$

$$\leq \mathcal{Q}_{k+1}(\tau-\tau_0)\exp[-\gamma(\tau-\tau_0)]$$

and

$$\mathcal{E}_k \leq c_k < \infty \tag{9.16}$$

on I, for all non-negative integers k.

Proof. By our construction we know the statement concerning (9.14) and (9.16) to be true for $k = 0, ..., m_d$ and the statement concerning (9.15) to be true for $k = 0, ..., m_d/2$. We want to prove the statement of the lemma by an inductive argument. Assume it to be true up to and including $k \geq m_d$ for (9.14) and (9.16) and to $k - m_d/2$ for (9.15). Let us introduce

$$\mathcal{F}_k = \frac{1}{2}\sum_{|\alpha|=k}\int_{T^d}|\nabla D^\alpha P|^2 d\theta$$

and

$$\mathcal{G}_k = \exp[-\gamma(\tau-\tau_0)]\mathcal{F}_k.$$

Our primary goal is to prove the following two inequalities

$$\frac{dE_{k+1}}{d\tau} \leq -2\gamma E_{k+1} + [\mathcal{Y}_{k+1,1} + \mathcal{Y}_{k+1,2}\mathcal{E}_{k+1}^{1/2}]\exp[-\gamma(\tau-\tau_0)]E_{k+1}^{1/2} \tag{9.17}$$

and

$$\frac{d\mathcal{E}_{k+1}}{d\tau} \leq [\mathcal{Z}_{k+1,1} + \mathcal{Z}_{k+1,2}E_{k+1}^{1/2} + \mathcal{Z}_{k+1,3}\mathcal{G}_k^{1/2}]\exp[-\gamma(\tau-\tau_0)]\mathcal{E}_{k+1}^{1/2} \tag{9.18}$$

where $\mathcal{Y}_{k+1,1}$, $\mathcal{Y}_{k+1,2}$, $\mathcal{Z}_{k+1,1}$, $\mathcal{Z}_{k+1,2}$ and $\mathcal{Z}_{k+1,3}$ are polynomials in $\tau-\tau_0$. Observe that we also have

$$\frac{d\mathcal{G}_k}{d\tau} \leq -\gamma\mathcal{G}_k + C_k\exp[-\gamma(\tau-\tau_0)/2]\mathcal{G}_k^{1/2}\mathcal{E}_{k+1}^{1/2}. \tag{9.19}$$

We have

$$\frac{dE_{k+1}}{d\tau} \leq -2\gamma E_{k+1} + \sum_{|\alpha|=k+1}\int_{T^d} f_\alpha D^\alpha \partial_\tau Q d\theta, \tag{9.20}$$

where

$$f_\alpha = f_\alpha(P,Q) = \partial_\tau(e^{2P}D^\alpha\partial_\tau Q) - \nabla\cdot(e^{2P-2\tau}\nabla D^\alpha Q)$$

and if $\alpha = \hat{\alpha} + e_l$,

$$f_\alpha = \partial_l f_{\hat{\alpha}} - 2\partial_l P f_{\hat{\alpha}} - \tag{9.21}$$

$$-2\partial_l\partial_\tau P e^{2P}D^{\hat{\alpha}}\partial_\tau Q + 2\nabla\partial_l P\cdot e^{2P-2\tau}\nabla D^{\hat{\alpha}}Q.$$

Let us first prove (9.17). Considering (9.17) and (9.20), it is enough to prove the estimate

$$\|e^{-P}f_\alpha\|_{L^2(T^d,\mathbb{R})} \leq [\Pi_{k+1,1} + \Pi_{k+1,2}\mathcal{E}_{k+1}^{1/2}]\exp[-\gamma(\tau-\tau_0)] \tag{9.22}$$

when $|\alpha| = k+1$, where $\Pi_{k+1,1}$ and $\Pi_{k+1,2}$ are polynomials. Since f_0 is zero, we inductively get the conclusion that f_α will contain two types of terms:

$$C_{\alpha_1,\alpha_2} e^{2P}(D^{\alpha_1}\partial_\tau P)(D^{\alpha_2}\partial_\tau Q) \tag{9.23}$$

and

$$B_{\alpha_1,\alpha_2} e^{2P-2\tau} \nabla D^{\alpha_1} P \cdot \nabla D^{\alpha_2} Q \tag{9.24}$$

where $|\alpha_1| \geq 1$ and $\alpha_1 + \alpha_2 = \alpha$. Consider terms of type (9.23). Considering (9.22), we wish to estimate

$$\|e^P (D^{\alpha_1}\partial_\tau P)(D^{\alpha_2}\partial_\tau Q)\|_{L^2(T^d,\mathbb{R})}.$$

If $|\alpha_1| \leq k - m_d/2$, then we can estimate $D^{\alpha_1}\partial_\tau P$ in the sup norm using Lemma 5.1 and the inductive hypothesis concerning (9.16). Since $|\alpha_2| \leq |\alpha| - 1 \leq k$, the inductive hypothesis concerning (9.14) yields the conclusion that

$$\|e^P (D^{\alpha_1}\partial_\tau P)(D^{\alpha_2}\partial_\tau Q)\|_{L^2(T^d,\mathbb{R})} \leq C_{k+1} \mathcal{P}_{|\alpha_2|} \exp[-\gamma(\tau - \tau_0)].$$

If $|\alpha_2| \leq k - m_d/2$, then we can take out $e^P D^{\alpha_2}\partial_\tau Q$ in the sup norm, using the inductive assumption concerning (9.15), in order to obtain

$$\|e^P (D^{\alpha_1}\partial_\tau P)(D^{\alpha_2}\partial_\tau Q)\|_{L^2(T^d,\mathbb{R})} \leq \sqrt{2} \mathcal{Q}_{|\alpha_2|+1} \exp[-\gamma(\tau - \tau_0)] \mathcal{E}_{|\alpha_1|}^{1/2}.$$

Regardless of whether $|\alpha_1| = k+1$ or not, we get an estimate that fits into (9.22). As we cannot have $|\alpha_2| > k - m_d/2$ and $|\alpha_1| > k - m_d/2$ at the same time, we have dealt with terms of the form (9.23). Terms of the form (9.24) can be handled similarly. Equation (9.17) follows.

As far as (9.18) is concerned, we have

$$\frac{d\mathcal{E}_{k+1}}{d\tau} \leq \sum_{|\alpha|=k+1} \int_{T^d} D^\alpha [e^{2P}(Q_\tau^2 - e^{-2\tau}|\nabla Q|^2)] D^\alpha \partial_\tau P d\theta. \tag{9.25}$$

Considering (9.18), it is thus enough to prove

$$\|D^\alpha [e^{2P}(Q_\tau^2 - e^{-2\tau}|\nabla Q|^2)]\|_{L^2(T^d,\mathbb{R})} \leq \tag{9.26}$$

$$\leq [\Pi_{k+1,1} + \Pi_{k+1,2} E_{k+1}^{1/2} + \Pi_{k+1,3} \mathcal{G}_k^{1/2}] \exp[-\gamma(\tau - \tau_0)]$$

for $|\alpha| = k+1$ and some polynomials $\Pi_{k+1,1}$, $\Pi_{k+1,2}$ and $\Pi_{k+1,3}$. Consider

$$D^\alpha (e^{2P} Q_\tau^2) = \sum_{l=0}^{k+1} \sum_{\alpha_1 + \ldots + \alpha_{l+2} = \alpha} C_{\alpha_1,\ldots,\alpha_{l+2}} e^{2P} D^{\alpha_1} P \cdots D^{\alpha_l} P D^{\alpha_{l+1}} \partial_\tau Q D^{\alpha_{l+2}} \partial_\tau Q. \tag{9.27}$$

Observe that we can bound $D^\beta P$ by a polynomial in supremum norm if $|\beta| \leq k - m_d/2$ and in L^2 norm if $|\beta| \leq k$ using the induction hypothesis concerning (9.16) and Lemma 5.1. We can also control $e^P D^\beta \partial_\tau Q$ in the sup norm, using (9.15) and the inductive assumption, if $|\beta| \leq k - m_d/2$, and in the L^2 norm if $|\beta| \leq k$. Consider a term in the sum (9.27). At most one $|\alpha_i|$ can be greater than $k - m_d/2$. If all α_i satisfy $|\alpha_i| \leq k$, then we get a bound

$$\|e^{2P} D^{\alpha_1} P \cdots D^{\alpha_l} P D^{\alpha_{l+1}} \partial_\tau Q D^{\alpha_{l+2}} \partial_\tau Q\|_{L^2(T^d,\mathbb{R})} \leq \Pi_{k+1} \exp[-2\gamma(\tau - \tau_0)]$$

where Π_{k+1} is a polynomial. If $|\alpha_i| = k+1$, and $i \leq l$, i.e. if $k+1$ derivatives hit one P, then

$$\|e^{2P} D^{\alpha_1} P \cdots D^{\alpha_l} P D^{\alpha_{l+1}} \partial_\tau Q D^{\alpha_{l+2}} \partial_\tau Q\|_{L^2(T^d,\mathbb{R})} \leq$$

$$\leq \Pi_{k+1} \exp[-2\gamma(\tau-\tau_0)] \|D^\alpha P\|_{L^2(T^d,\mathbb{R})} \leq \Pi'_{k+1} \exp[-3\gamma(\tau-\tau_0)/2] \mathcal{G}_k^{1/2}.$$

Finally, if one $|\alpha_i| = k+1$, and $i > l$, i.e. if all the derivatives hit one $\partial_\tau Q$, then

$$\|e^{2P} D^{\alpha_1} P \cdots D^{\alpha_l} P D^{\alpha_{l+1}} \partial_\tau Q D^{\alpha_{l+2}} \partial_\tau Q\|_{L^2(T^d,\mathbb{R})} \leq$$

$$\leq \Pi_{k+1} \exp[-\gamma(\tau-\tau_0)] \|e^P D^\alpha \partial_\tau Q\|_{L^2(T^d,\mathbb{R})} \leq \Pi'_{k+1} \exp[-\gamma(\tau-\tau_0)] E_{k+1}^{1/2}.$$

As the argument for $D^\alpha(e^{2P-2\tau}|\nabla P|^2)$ is similar (9.26), and thereby (9.18), follows. Let

$$\mathcal{H}_k = \mathcal{E}_{k+1} + E_{k+1} + \mathcal{G}_k.$$

The estimates (9.17)-(9.19) imply

$$\frac{d\mathcal{H}_k}{d\tau} \leq \mathcal{W}_{k,1} \exp[-\gamma(\tau-\tau_0)/2] \mathcal{H}_k^{1/2} + \mathcal{W}_{k,2} \exp[-\gamma(\tau-\tau_0)/2] \mathcal{H}_k, \tag{9.28}$$

where $\mathcal{W}_{k,1}$ and $\mathcal{W}_{k,2}$ are polynomials. This inequality implies that $\mathcal{H}_k$ is bounded on I. In fact, since $a^{1/2} \leq \frac{1}{2}(1+a) \leq 1+a$ for $a \geq 0$, (9.28) yields

$$\frac{d(1+\mathcal{H}_k)}{d\tau} \leq \{\mathcal{W}_{k,1} \exp[-\gamma(\tau-\tau_0)/2] + \mathcal{W}_{k,2} \exp[-\gamma(\tau-\tau_0)/2]\}(1+\mathcal{H}_k)$$

whence $1+\mathcal{H}_k$ is bounded. Thus $\mathcal{E}_{k+1}$ is bounded, yielding (9.16) for $k+1$, which when inserted in (9.17) implies (9.14) for $k+1$, which, together with the induction hypothesis yields (9.15) for $k+1-m_d/2$ by Sobolev embedding. □

Observe that using (9.16) and Lemma 5.1, we can conclude that

$$|D^\alpha P| \leq \mathcal{R}_\alpha$$

on $I \times T^d$ for all α, where $\mathcal{R}_\alpha$ are first degree polynomials in $\tau - \tau_0$. Using this together with (2.1) and (9.15), we conclude that

$$|D^\alpha \partial_\tau^2 P| \leq \Gamma_\alpha \exp[-2\gamma(\tau-\tau_0)]$$

on $I \times T^d$ for all α, where Γ_α are polynomials in $\tau - \tau_0$. This implies that $\partial_\tau P$ converges in any $C^k(T^d,\mathbb{R})$ norm as $\tau \to \infty$ to a smooth function v on T^d, and that (9.6) holds. Equation (9.6) in its turn implies that

$$P(\tau,\cdot) - v \cdot (\tau-\tau_0) \to w$$

in any $C^k(T^d,\mathbb{R})$, and that we have (9.7). Observe now that by (2.2) we have

$$\partial_\tau(e^{2P}\partial_\tau Q) = \nabla \cdot (e^{2P-2\tau}\nabla Q).$$

Thus

$$|\partial_\tau D^\alpha(e^{2P}Q_\tau)| = |\nabla \cdot D^\alpha(e^{2P-2\tau}\nabla Q)| \leq \pi_\alpha \exp[-2\gamma(\tau-\tau_0)]$$

combining (9.15) with the fact that $P - \tau \leq C - \gamma(\tau-\tau_0)$. Here and below, π_α will denote a polynomial in $\tau - \tau_0$. This inequality can be integrated in order to yield the conclusion that there is an $r \in C^\infty(T^d,\mathbb{R})$ such that

$$|D^\alpha(e^{2P}Q_\tau - r)| \leq \pi_\alpha \exp[-2\gamma(\tau-\tau_0)]. \tag{9.29}$$

Let now

$$\rho = v \cdot (\tau-\tau_0) + w,$$

and observe that we have (9.7). We would now like to replace P in (9.29) with ρ. Consider for that reason

$$D^\alpha(e^{2\rho}Q_\tau - r) = D^\alpha[(e^{2\rho} - e^{2P})Q_\tau] + D^\alpha(e^{2P}Q_\tau - r).$$

The second term we can bound using (9.29). Let us consider the first term,

$$D^\alpha[(e^{2\rho}-e^{2P})Q_\tau]=\sum_{\alpha_1+\alpha_2=\alpha}C_{\alpha_1,\alpha_2}D^{\alpha_1}(e^{2\rho-2P}-1)D^{\alpha_2}(e^{2P}Q_\tau).$$

The second factor on the right hand side is bounded by a constant for $\tau\geq\tau_0$ by the above, and so the factor of interest is

$$D^{\alpha_1}(e^{2\rho-2P}-1).$$

Consider first the case $\alpha_1=0$. If τ is great enough, then $|2\rho-2P|\leq 1$, so that

$$|e^{2\rho-2P}-1|=|\sum_{n=1}^{\infty}\frac{(2\rho-2P)^n}{n!}|\leq 2|\rho-P|\exp[|2\rho-2P|]\leq 2e|\rho-P|\leq$$

$$\leq\pi_0\exp[-2\gamma(\tau-\tau_0)],$$

where we have used (9.7) in the last inequality. If $|\alpha_1|\geq 1$ we get similar estimates for less complicated reasons. To sum up,

$$|D^\alpha(e^{2\rho}Q_\tau-r)|\leq\pi_\alpha\exp[-2\gamma(\tau-\tau_0)],$$

proving (9.10). We conclude that (9.8) holds, and therefore there is a $q\in C^\infty(T^d,\mathbb{R})$ such that (9.9) holds. Let us compute

$$e^{2\rho}(Q-q)=e^{2\rho}(-\int_\tau^\infty Q_\tau(s)ds)=-\int_\tau^\infty e^{2\rho(\tau)}Q_\tau(s)ds=$$

$$=-\int_\tau^\infty e^{2\rho(\tau)-2\rho(s)}(e^{2\rho(s)}Q_\tau(s)-r)ds-\int_\tau^\infty e^{2\rho(\tau)-2\rho(s)}rds.$$

Observe that r is independent of s and compute

$$\int_\tau^\infty e^{2\rho(\tau)-2\rho(s)}ds=\int_\tau^\infty e^{-2v\cdot(s-\tau)}ds=\frac{1}{2v}.$$

We thus conclude that

$$D^\alpha(e^{2\rho}(Q-q)+\frac{r}{2v})=-D^\alpha\left(\int_\tau^\infty e^{2\rho(\tau)-2\rho(s)}(e^{2\rho(s)}Q_\tau(s)-r)ds\right)=$$

$$=-\sum_{\alpha_1+\alpha_2=\alpha}^{k}C_{\alpha_1,\alpha_2}\left(\int_\tau^\infty D^{\alpha_1}[e^{2\rho(\tau)-2\rho(s)}]D^{\alpha_2}(e^{2\rho(s)}Q_\tau(s)-r)ds\right).$$

In order to estimate this expression, consider

$$|\int_\tau^\infty D^{\alpha_1}[e^{2\rho(\tau)-2\rho(s)}]D^{\alpha_2}(e^{2\rho(s)}Q_\tau(s)-r)ds|\leq$$

$$\leq\left(\int_\tau^\infty[D^{\alpha_1}e^{2\rho(\tau)-2\rho(s)}]^2ds\right)^{1/2}\left(\int_\tau^\infty[D^{\alpha_2}(e^{2\rho(s)}Q_\tau(s)-r)]^2ds\right)^{1/2}$$

The integrand in the first factor can be bounded by a polynomial in $s-\tau$ multiplied by $\exp[-4v\cdot(s-\tau)]$. In consequence the first factor is bounded by a constant. The integrand appearing in the second factor can be bounded by

$$\pi_{\alpha_2}\exp[-4\gamma(s-\tau_0)]$$

where π_{α_2} is a polynomial in $s-\tau_0$. Adding up these observations, we get the conclusion

$$|D^\alpha(e^{2\rho}(Q-q)+\frac{r}{2v})| \leq \pi_\alpha \exp[-2\gamma(\tau-\tau_0)],$$

proving (9.11).

Let us now prove that solutions with the desired asymptotics satisfy the initial conditions at late enough times. Due to (9.12) there is a $w \in C^{m_d+1}(T^d,\mathbb{R})$ such that

$$\|P(\tau,\cdot)-v\cdot(\tau-\tau_0)-w\|_{C^{m_d+1}(T^d,\mathbb{R})} \to 0 \tag{9.30}$$

as $\tau \to \infty$. As a consequence, the $\epsilon_k'(\tau)$ are bounded for the entire future and the $\nu_k'(\tau)$ do not grow faster than linearly. Furthermore, by (9.30) and (9.12),

$$\sum_{|\alpha|\leq m_d+1} |D^\alpha \partial_\tau Q| \leq C\exp(-v\cdot\tau-\epsilon\tau)$$

Thus there is a $q \in C^{m_d+1}(T^d,\mathbb{R})$ such that

$$\sum_{|\alpha|\leq m_d+1} \|e^P D^\alpha(Q-q)\|_{C(T^d,\mathbb{R})} \leq Ce^{-\epsilon\tau}.$$

Since $P \leq C+(1-\gamma)\tau$ and $\gamma>0$, we get the conclusion that $e_k'(\tau)$ decays to zero exponentially in time. The last statement of the theorem follows. □

10. Curvature blow up

Let us make some observations concerning the geometry of the metric (1.1) given the conclusions of the previous section.

Proposition 10.1. *Consider a metric of the form (1.1). Assuming P and Q have the asymptotic behaviour obtained as a conclusion in Theorem 9.1, we have*

$$\lim_{\tau\to\infty} \inf_{\theta\in S^1} |(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})(\tau,\theta)| = \infty.$$

Proof. We proceed as in [8]. Consider the orthonormal basis given by

$$e_0 = e^{\lambda/4+3\tau/4}\partial_\tau,\quad e_1 = e^{\lambda/4-\tau/4}\partial_\theta,\quad e_2 = e^{\tau/2-P/2}\partial_\sigma,\quad e_3 = e^{\tau/2+P/2}(\partial_\delta - Q\partial_\sigma).$$

There is a natural scaling given by

$$\phi = e^{-\lambda/4-3\tau/4}.$$

Only few terms in $\phi^2 R_{\alpha\beta\gamma\delta}$, where we assume that the indexes are with respect to the orthonormal basis mentioned above, are non-negligible. This makes the computation of the Kretschmann scalar manageable and in fact,

$$\phi^4 R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$$

converges to a smooth non-zero function on S^1. Since λ equals $v^2(\theta)\tau$ up to something bounded, we get the conclusion of the proposition. □

There is also an elementary proof of causal geodesic incompleteness in our situation.

Proposition 10.2. *Consider an inextendible causal geodesic $\gamma : (s_-, s_+) \to M$, where $M = \mathbb{R} \times T^3$ with a metric of the form (1.1) where P and Q have the asymptotic behaviour obtained as a conclusion in Theorem 9.1. Assume furthermore that $< \gamma'(s), \partial_\tau|_{\gamma(s)} >< 0$. Then γ is future incomplete and*

$$\lim_{s \to s_+ -} |(R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta})[\gamma(s)]| = \infty.$$

Proof. Let the basis e_μ be as in the proof of the previous proposition and define

$$f_0 = - < \gamma', e_0|_\gamma >, \quad f_k =< \gamma', e_k|_\gamma >$$

for $k = 1, 2, 3$. Observe that $\sum f_k^2 \leq f_0^2$ due to causality. Furthermore, if we let the τ component of γ be denoted by γ_0, then $d\gamma_0/ds > 0$. Thus, if γ_0 is bounded from above, it converges to a finite value as $s \to s_+ -$. By the causality of the curve and compactness of the spatial slices, we then get the conclusion that γ converges. Thus γ is continuously extendible, leading to the conclusion that it is extendible considered as a geodesic. This contradicts our assumptions, and we conclude that $\gamma_0(s) \to \infty$ as $s \to s_+ -$. Consider

$$\frac{df_0}{ds} = - < \gamma', \nabla_{\gamma'} e_0 >= - \sum_{\mu,\nu} f_\mu f_\nu < e_\mu, \nabla_{e_\nu} e_0 > \circ \gamma.$$

Let ϕ be as in the previous proposition. Using the assumptions concerning the asymptotics one sees that

$$\phi < e_1, \nabla_{e_1} e_0 >= -\frac{1}{4}(\lambda_\tau - 1), \quad \phi < e_2, \nabla_{e_2} e_0 >= -\frac{1}{2}(1 - P_\tau)$$

and

$$\phi < e_3, \nabla_{e_3} e_0 >= -\frac{1}{2}(1 + P_\tau)$$

and that all other elements of the matrix $\phi < e_\mu, \nabla_{e_\nu} e_0 >$ converge to zero exponentially with τ. Letting

$$\theta_k = \phi \circ \gamma < e_k, \nabla_{e_k} e_0 > \circ \gamma,$$

we have

$$\frac{df_0}{ds} = -\psi \circ \gamma \sum_k f_k^2 \theta_k + \psi \circ \gamma \delta_1 f_0^2 \geq$$

$$\geq \psi \circ \gamma f_0^2 \frac{1}{4}(\lambda_\tau \circ \gamma - 1) + \psi \circ \gamma \delta_1 f_0^2,$$

if s is close enough to s_+, where $\psi = 1/\phi$ and $\delta_1(s) \to 0$ as $s \to s_+ -$ (observe that $\theta_1 > 0$ and $\theta_2, \theta_3 < 0$ if s is close enough to s_+). Compute

$$\frac{d\psi \circ \gamma}{ds} = \frac{\partial \psi}{\partial \tau} \circ \gamma \frac{d\gamma_0}{ds} + \frac{\partial \psi}{\partial \theta} \circ \gamma \frac{d\gamma_1}{ds}$$

where γ_1 is the θ-coordinate of γ (observe that even though this is not well defined, the derivative is). However,

$$\frac{d\gamma_0}{ds} = \psi \circ \gamma f_0, \quad \frac{d\gamma_1}{ds} = \exp[\lambda \circ \gamma/4 - \gamma_0/4] f_1$$

so that

$$\frac{d\psi \circ \gamma}{ds} = \psi^2 \circ \gamma f_0 [\frac{1}{4}(\lambda_\tau \circ \gamma + 3) + \delta_2]$$

where $\delta_2(s) \to 0$ as $s \to s_+-$. Letting $g = f_0 \cdot \psi \circ \gamma$, we get

$$\frac{dg}{ds} = \frac{df_0}{ds}\psi \circ \gamma + f_0 \frac{d\psi \circ \gamma}{ds} \geq g^2[\frac{1}{4}(\lambda_\tau \circ \gamma - 1) + \delta_1 + \frac{1}{4}(\lambda_\tau \circ \gamma + 3) + \delta_2].$$

Thus there is an s_1 such that for $s \geq s_1$

$$\frac{dg}{ds}(s) \geq \frac{1}{2} g^2(s),$$

since λ_τ is bounded from below by a positive constant for large τ. We thus get the conclusion that the geodesic is future incomplete. Since $\gamma_0(s) \to \infty$ as $s \to s_+-$, the statement concerning curvature blow up follows from the previous proposition. □

11. Conclusions and observations

This paper provides a proof of the statement that an open set of initial data yields asymptotics of the form (1.6) and (1.7). The fact that we have a condition on initial data will hopefully be useful when trying to make further progress in understanding Gowdy spacetimes. However, the method of proof relies heavily on the fact that the non-linear terms in the equations die out exponentially. This will not be the case in general and different methods will have to be developed. Some intuition for what should happen has been developed in [2], but how to make these ideas rigorous is far from clear. Since approaching the general problem is difficult, it is natural to try to find an easier problem which can be used as a model for some of the dynamics. One way of obtaining a model problem is to carry out intuitive arguments similar to those in [2]. For example, if $P_\tau > 1$, $e^P Q_\tau$ should be negligible and Q_θ should be a time independent function. Inserting these assumptions in (1.2) one obtains

$$P_{\tau\tau} - e^{-2\tau} P_{\theta\theta} = -e^{2P-2\tau} Q_\theta^2.$$

Replacing Q_θ with 1 and calling $P - \tau$ P, one gets the equation

$$P_{\tau\tau} - e^{-2\tau} P_{\theta\theta} = -e^{2P}. \tag{11.1}$$

This equation would, in the terminology of [2], model a bounce. Considering an arbitrary solution to (11.1), P_τ should converge exponentially to a smooth negative function on S^1. One question of interest would then be to find out how fast this process of convergence occurs. This is an example of a model problem which is certainly easier to handle, but which still contains some of the dynamics of the real situation. Since most of the results concerning the asymptotic behaviour of Einstein's equations in the spatially inhomogeneous case make assumptions that exclude the possibility of the non-linear terms being of any importance, the study of even a simple model problem where this is not the case should be of interest.

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